

Alternating Runtime and Size Complexity Analysis of Integer Programs

Jürgen Giesl

LuFG Informatik 2, RWTH Aachen University, Germany

joint work with [M. Brockschmidt](#), [F. Emmes](#), [S. Falke](#), and [C. Fuhs](#)

Complexity Analysis for Integer Programs

- Termination analysis of imperative programs: polynomial rank functions

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**
- **Problem:** complexity from **combination** of polynomial rank functions

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**
- **Problem:** complexity from **combination** of polynomial rank functions

```
while  $i > 0$  do  
     $i = i - 1$   
  
done  
  
while  $x > 0$  do  
     $x = x - 1$   
done
```

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**
- **Problem:** complexity from **combination** of polynomial rank functions

Termination: lexicographic combination of

$$f_1(x, i) = i$$

$$f_2(x, i) = x$$

```
while i > 0 do
```

```
  i = i - 1
```

```
done
```

```
while x > 0 do
```

```
  x = x - 1
```

```
done
```

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**
- **Problem:** complexity from **combination** of polynomial rank functions

Termination: lexicographic combination of

$$f_1(x, i) = i$$

$$f_2(x, i) = x$$

Complexity: linear

```
while i > 0 do
```

```
  i = i - 1
```

```
done
```

```
while x > 0 do
```

```
  x = x - 1
```

```
done
```

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**
- **Problem:** complexity from **combination** of polynomial rank functions

Termination: lexicographic combination of

$$f_1(x, i) = i$$

$$f_2(x, i) = x$$

Complexity: linear

```
while i > 0 do
```

```
  i = i - 1
```

```
  x = x + i
```

```
done
```

```
while x > 0 do
```

```
  x = x - 1
```

```
done
```

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**
- **Problem:** complexity from **combination** of polynomial rank functions

Termination: lexicographic combination of

$$f_1(x, i) = i$$

$$f_2(x, i) = x$$

Complexity: quadratic

```
while i > 0 do
```

```
  i = i - 1
```

```
  x = x + i
```

```
done
```

```
while x > 0 do
```

```
  x = x - 1
```

```
done
```

Complexity Analysis for Integer Programs

- **Termination analysis** of imperative programs: polynomial rank functions
- **Goal:** use polynomial rank functions for **complexity analysis**
- **Problem:** complexity from **combination** of polynomial rank functions

Termination: lexicographic combination of

$$f_1(x, i) = i$$

$$f_2(x, i) = x$$

Complexity: quadratic

```
while i > 0 do
```

```
  i = i - 1
```

```
  x = x + i
```

```
done
```

```
while x > 0 do
```

```
  x = x - 1
```

```
done
```

- **Solution:** alternate between finding **runtime** and **size** bounds

Runtime and Size Approximation Function

Input: List x

ℓ_0 : List y = null

ℓ_1 : **while** x $\neq$ null **do**

 y = **new** List(x.val, y)

 x = x.next

done

List z = y

ℓ_2 : **while** z $\neq$ null **do**

 List u = z.next

ℓ_3 : **while** u $\neq$ null **do**

 z.val += u.val

 u = u.next

done

z = z.next

done

Runtime and Size Approximation Function

Input: List x

ℓ_0 : List y = null

ℓ_1 : **while** x $\neq$ null **do**

 y = **new** List(x.val, y)

 x = x.next

done

List z = y

ℓ_2 : **while** z $\neq$ null **do**

 List u = z.next

ℓ_3 : **while** u $\neq$ null **do**

 z.val += u.val

 u = u.next

done

 z = z.next

done

$y = [5, 1, 3] \quad \rightsquigarrow \quad y = [5 + 1 + 3, 1 + 3, 3]$

Runtime and Size Approximation Function

Input: List x

ℓ_0 : List y = null

ℓ_1 : **while** x $\neq$ null **do**

 y = **new** List(x.val, y)

 x = x.next

done

List z = y

ℓ_2 : **while** z $\neq$ null **do**

 List u = z.next

ℓ_3 : **while** u $\neq$ null **do**

 z.val += u.val

 u = u.next

done

z = z.next

done

Integer
abstraction



Runtime and Size Approximation Function

Input: List x

ℓ_0 : List $y = \text{null}$

ℓ_1 : **while** $x \neq \text{null}$ **do**
 $y = \text{new List}(x.\text{val}, y)$
 $x = x.\text{next}$

done

List $z = y$

ℓ_2 : **while** $z \neq \text{null}$ **do**
 List $u = z.\text{next}$

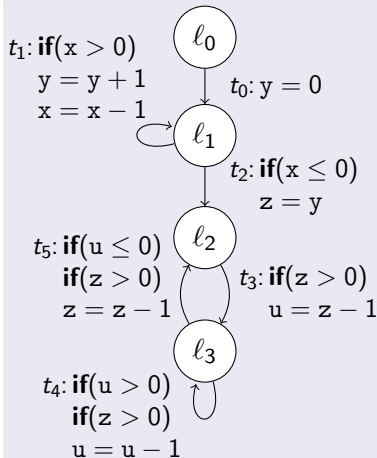
ℓ_3 : **while** $u \neq \text{null}$ **do**
 $z.\text{val} += u.\text{val}$
 $u = u.\text{next}$

done

$z = z.\text{next}$

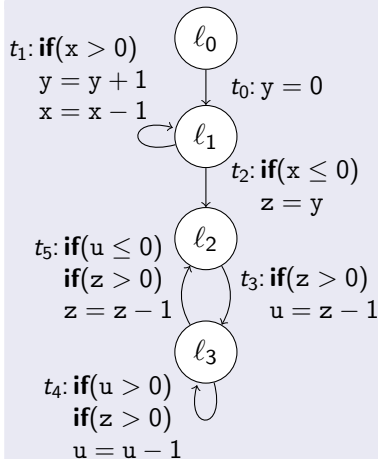
done

Integer
abstraction
 $\Rightarrow$



Runtime and Size Approximation Function

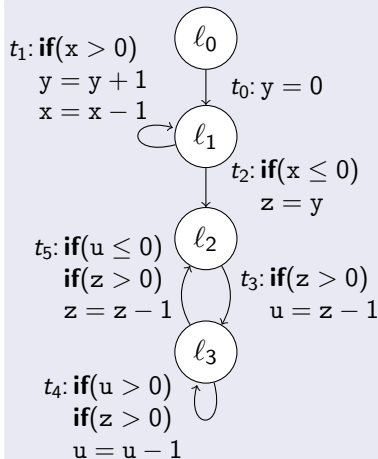
Goal: find complexity bounds w.r.t.
the *sizes* of the input variables



Runtime and Size Approximation Function

Goal: find complexity bounds w.r.t.
the *sizes* of the input variables

- **Runtime approximation function $\mathcal{R}(t)$:**
bound on number of times that
transition t occurs in executions



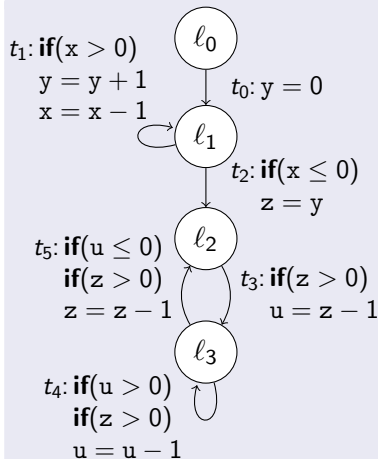
Runtime and Size Approximation Function

Goal: find complexity bounds w.r.t.
the *sizes* of the input variables

- Runtime approximation function $\mathcal{R}(t)$:

bound on number of times that
transition t occurs in executions

e.g. $\mathcal{R}(t_1) = |x|$,



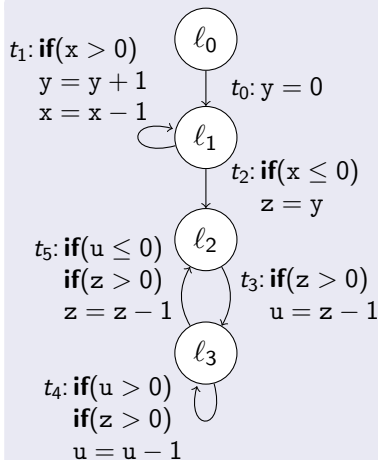
Runtime and Size Approximation Function

Goal: find complexity bounds w.r.t.
the *sizes* of the input variables

- Runtime approximation function $\mathcal{R}(t)$:

bound on number of times that
transition t occurs in executions

e.g. $\mathcal{R}(t_1) = |x|$,
 $\mathcal{R}(t_4) = |x| + x^2$



Runtime and Size Approximation Function

Goal: find complexity bounds w.r.t.
the *sizes* of the input variables

- **Runtime approximation function $\mathcal{R}(t)$:**

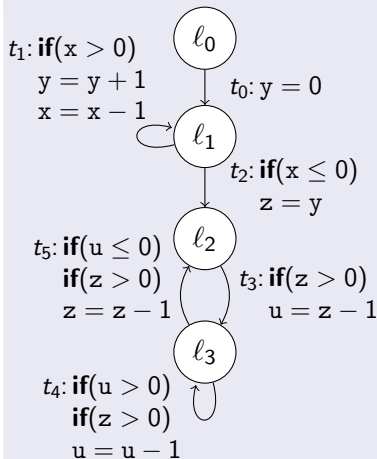
bound on number of times that
transition t occurs in executions

e.g. $\mathcal{R}(t_1) = |x|$,

$\mathcal{R}(t_4) = |x| + x^2$

- **Size approximation function $\mathcal{S}(t, v')$:**

bound on $|v|$ after using transition t
in program executions



Runtime and Size Approximation Function

Goal: find complexity bounds w.r.t.
the *sizes* of the input variables

- **Runtime approximation function $\mathcal{R}(t)$:**

bound on number of times that
transition t occurs in executions

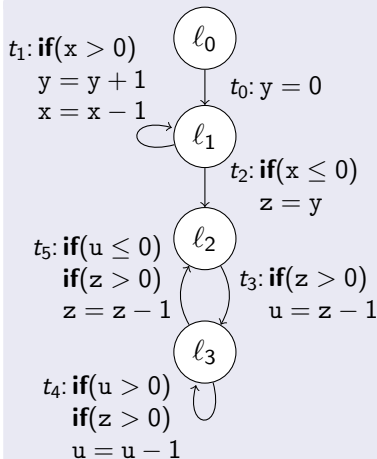
e.g. $\mathcal{R}(t_1) = |x|$,

$\mathcal{R}(t_4) = |x| + x^2$

- **Size approximation function $\mathcal{S}(t, v')$:**

bound on $|v|$ after using transition t
in program executions

e.g. $\mathcal{S}(t_1, y') = |x|$



Runtime and Size Approximation Function

Goal: find complexity bounds w.r.t.
the *sizes* of the input variables

- **Runtime approximation function $\mathcal{R}(t)$:**

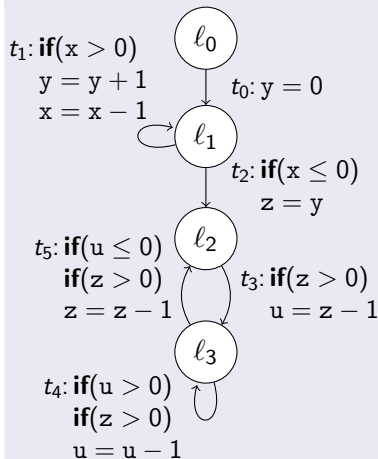
bound on number of times that
transition t occurs in executions

e.g. $\mathcal{R}(t_1) = |x|$,
 $\mathcal{R}(t_4) = |x| + x^2$

- **Size approximation function $\mathcal{S}(t, v')$:**

bound on $|v|$ after using transition t
in program executions

e.g. $\mathcal{S}(t_1, y') = |x|$

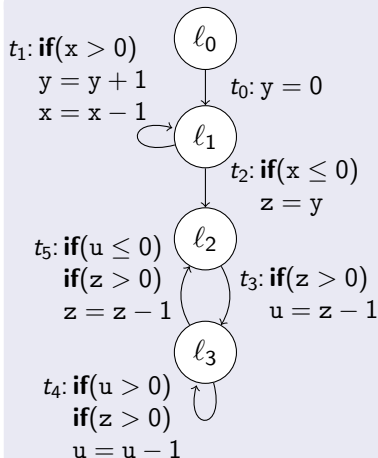


Overall runtime is bounded by $\mathcal{R}(t_1) + \dots + \mathcal{R}(t_5) = 3 + 4 \cdot |x| + x^2$.

Runtime Bounds from Polynomial Rank Functions

Initial approximations

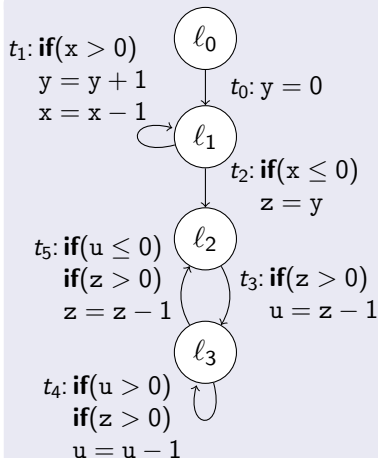
- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0



Runtime Bounds from Polynomial Rank Functions

Initial approximations

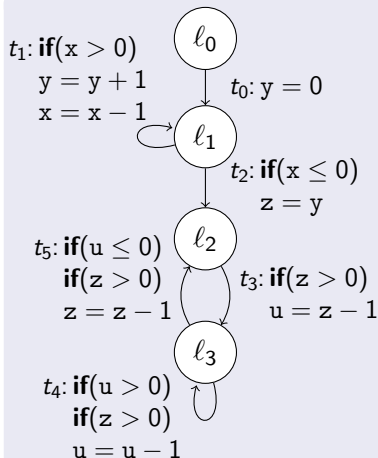
- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t



Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $S(t, v') = ?$ for all t and variables v

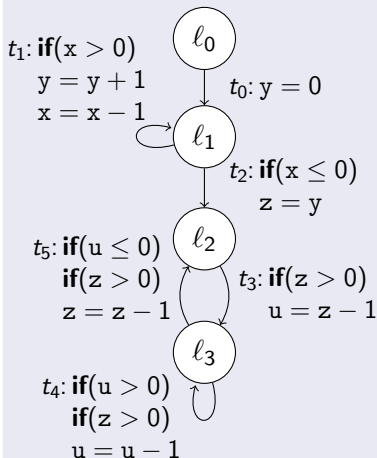


Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$



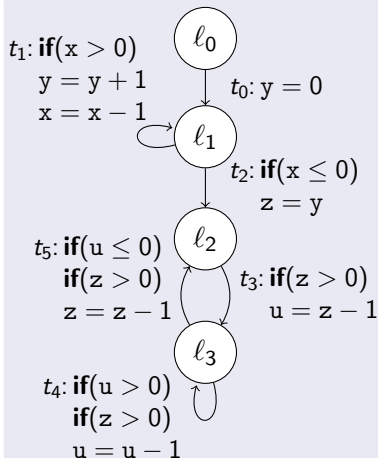
Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$



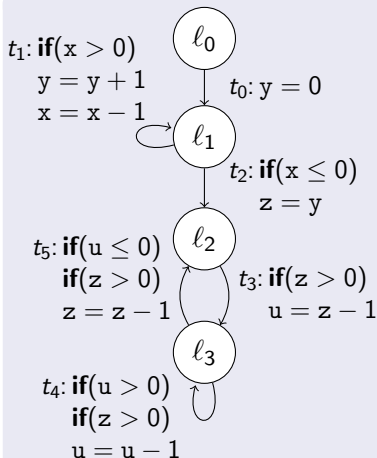
Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$



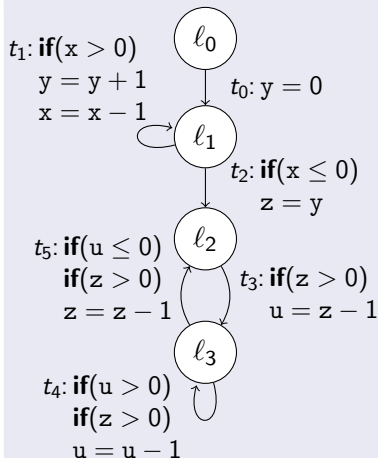
Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_> \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$



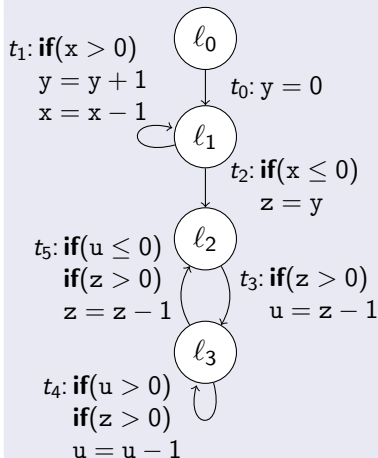
Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_> \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n),$
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq 1$



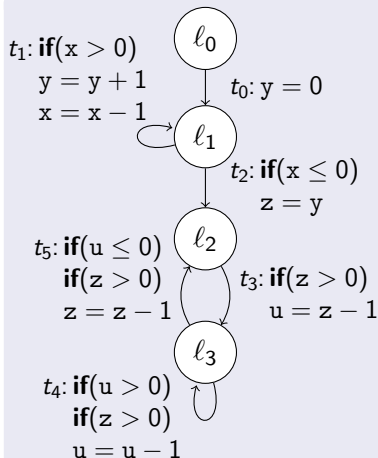
Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_\prec \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n),$
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq 1$
- $\mathcal{Pol}(\ell) = x$ for all locations ℓ



Runtime Bounds from Polynomial Rank Functions

Initial approximations

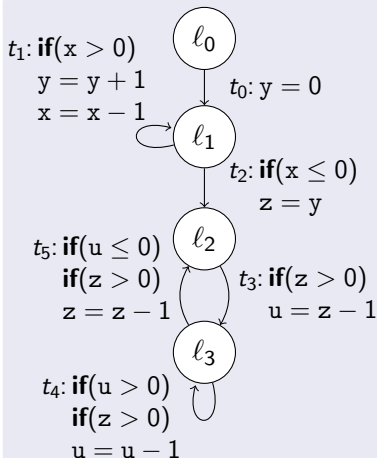
- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_> \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$,
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq 1$

• $\mathcal{Pol}(\ell) = x$ for all locations ℓ

• Transition $t_1: (\ell_1, \quad x > 0 \wedge y' = y + 1 \wedge x' = x - 1, \quad \ell_1)$



Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

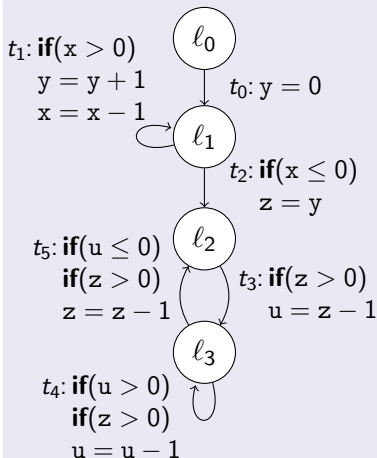
Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_> \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$,
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq 1$

• $\mathcal{Pol}(\ell) = x$ for all locations ℓ

• Transition t_1 : $(\ell_1, \quad x > 0 \wedge y' = y + 1 \wedge x' = x - 1, \quad \ell_1)$

• $t_1 \in \mathcal{T}_>$



Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

Polynomial rank function (PRF) for program $\mathcal{T}$

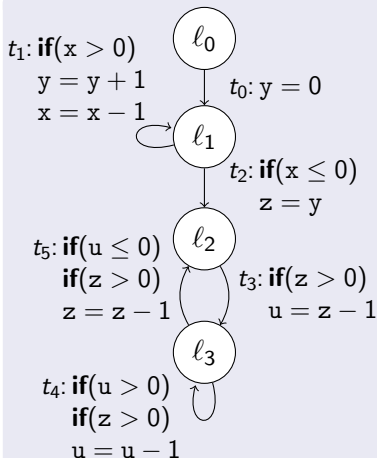
- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_> \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n),$
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq 1$

• $\mathcal{Pol}(\ell) = x$ for all locations ℓ

• Transition t_1 : $(\ell_1, \quad x > 0 \wedge y' = y + 1 \wedge x' = x - 1, \quad \ell_1)$

• $t_1 \in \mathcal{T}_>$, since

$$x > x'$$



Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

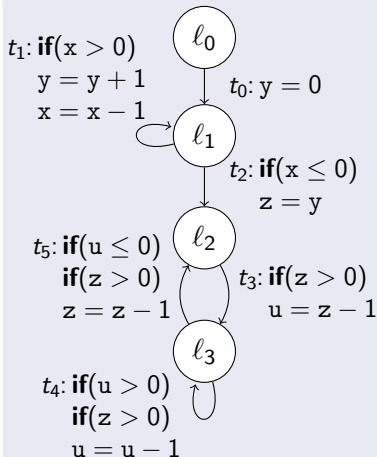
Polynomial rank function (PRF) for program $\mathcal{T}$

- $\mathcal{Pol}$ maps locations to $\mathbb{Z}[v_1, \dots, v_n]$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_> \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n),$
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq 1$

• $\mathcal{Pol}(\ell) = x$ for all locations ℓ

• Transition t_1 : $(\ell_1, \quad x > 0 \wedge y' = y + 1 \wedge x' = x - 1, \quad \ell_1)$

• $t_1 \in \mathcal{T}_>$, since $x > 0 \wedge y' = y + 1 \wedge x' = x - 1 \Rightarrow x > x'$



Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1$ as t_0 starts in initial location ℓ_0
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

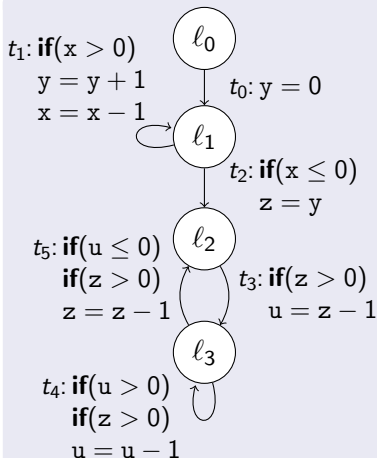
Polynomial rank function (PRF) for program $\mathcal{T}$

- for all $t \in \mathcal{T}_\succ$, set $\mathcal{R}(t) := |\text{Pol}(\ell_0)|$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\text{Pol}(\ell))(v_1, \dots, v_n) \geq (\text{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_\succ \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\text{Pol}(\ell))(v_1, \dots, v_n) > (\text{Pol}(\ell'))(v'_1, \dots, v'_n)$,
 $\tau \Rightarrow (\text{Pol}(\ell))(v_1, \dots, v_n) \geq 1$

• $\text{Pol}(\ell) = x$ for all locations ℓ

• Transition t_1 : $(\ell_1, \quad x > 0 \wedge y' = y + 1 \wedge x' = x - 1, \quad \ell_1)$

• $t_1 \in \mathcal{T}_\succ$, since $x > 0 \wedge y' = y + 1 \wedge x' = x - 1 \Rightarrow x > x'$



Runtime Bounds from Polynomial Rank Functions

Initial approximations

- $\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

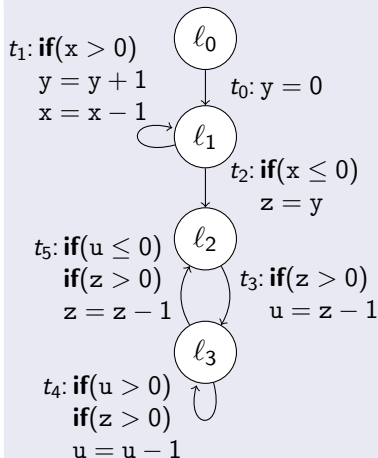
Polynomial rank function (PRF) for program $\mathcal{T}$

- for all $t \in \mathcal{T}_\succ$, set $\mathcal{R}(t) := |\mathcal{Pol}(\ell_0)|$
- for all $(\ell, \tau, \ell') \in \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n)$
- for all $(\ell, \tau, \ell') \in \mathcal{T}_\succ \subseteq \mathcal{T}$:
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) > (\mathcal{Pol}(\ell'))(v'_1, \dots, v'_n),$
 $\tau \Rightarrow (\mathcal{Pol}(\ell))(v_1, \dots, v_n) \geq 1$

• $\mathcal{Pol}(\ell) = x$ for all locations ℓ

• Transition t_1 : $(\ell_1, \quad x > 0 \wedge y' = y + 1 \wedge x' = x - 1, \quad \ell_1)$

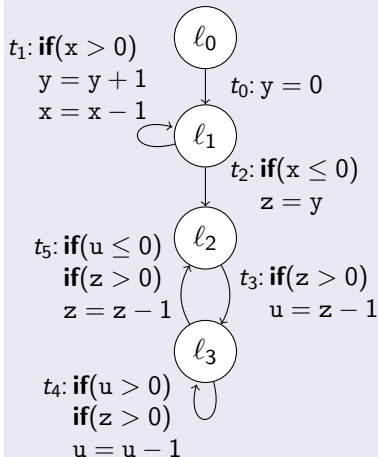
• $t_1 \in \mathcal{T}_\succ$, since $x > 0 \wedge y' = y + 1 \wedge x' = x - 1 \Rightarrow x > x'$



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

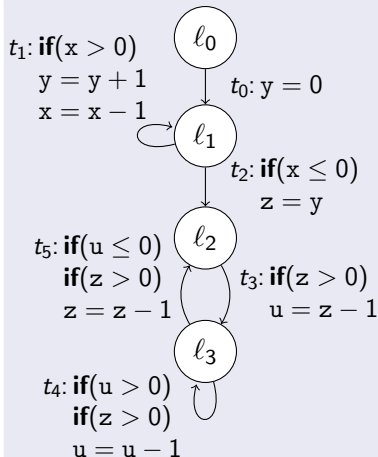
- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

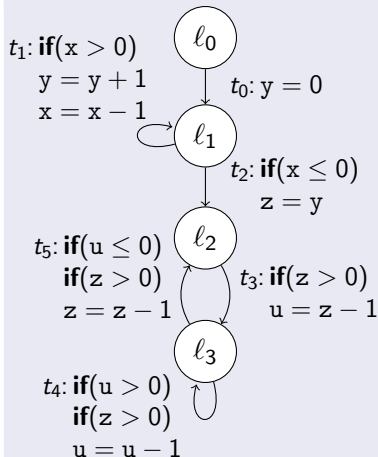
- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
 - $\mathcal{R}(t) = ?$ for all other transitions t
 - $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset**
 $\mathcal{T}' = \{t_1, \dots, t_5\}$



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

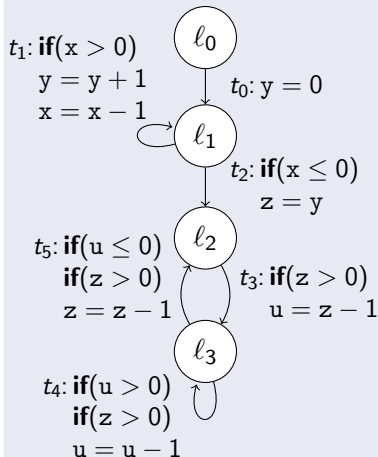
- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\text{Pol}(\ell_1) = 1$, $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = 0$



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

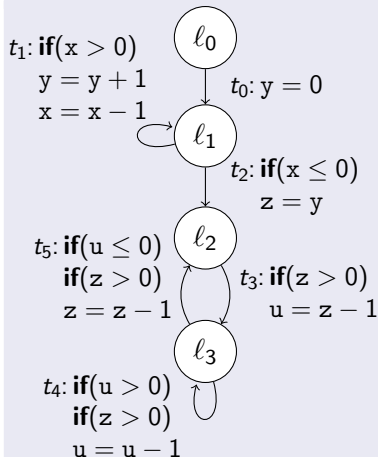
- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\text{Pol}(\ell_1) = 1$, $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = 0$
- $t_2 \in \mathcal{T}'_{\gamma}$



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

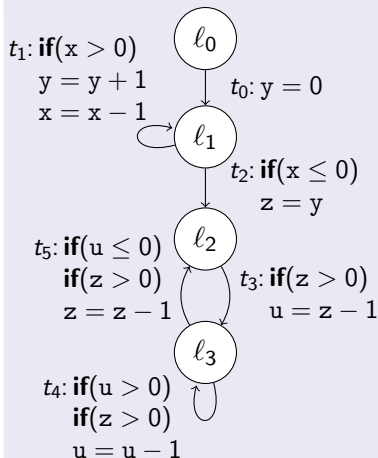
- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\text{Pol}(\ell_1) = 1$, $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = 0$
- $t_2 \in \mathcal{T}'_{\prec}$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_1 , t_2 is used at most $|\text{Pol}(\ell_1)| = 1$ times.



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

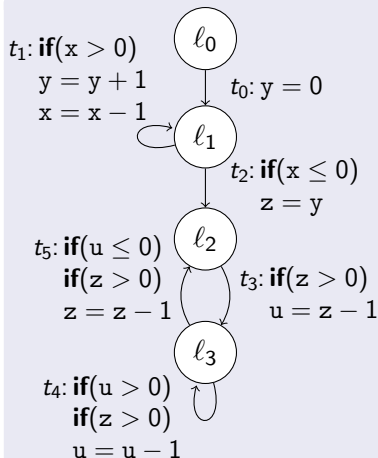
- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\mathcal{Pol}(\ell_1) = 1$, $\mathcal{Pol}(\ell_2) = \mathcal{Pol}(\ell_3) = 0$
- $t_2 \in \mathcal{T}'_{\prec}$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_1 , t_2 is used at most $|\mathcal{Pol}(\ell_1)| = 1$ times.
- For global result:
consider how often $\mathcal{T}'$ is reached (by t_0)



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\mathcal{Pol}(\ell_1) = 1$, $\mathcal{Pol}(\ell_2) = \mathcal{Pol}(\ell_3) = 0$
- $t_2 \in \mathcal{T}'_{\prec}$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_1 , t_2 is used at most $|\mathcal{Pol}(\ell_1)| = 1$ times.
- For global result:
consider how often $\mathcal{T}'$ is reached (by t_0)

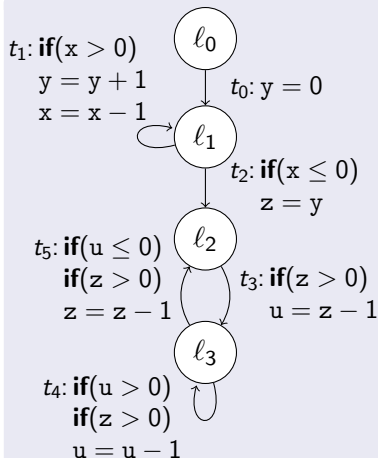


$\Rightarrow$ multiply t_0 's runtime approximation $\mathcal{R}(t_0)$ with local bound $|\mathcal{Pol}(\ell_1)|$

Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\text{Pol}(\ell_1) = 1$, $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = 0$
- $t_2 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_1 , t_2 is used at most $|\text{Pol}(\ell_1)| = 1$ times.
- For global result:
consider how often $\mathcal{T}'$ is reached (by t_0)



$\Rightarrow$ multiply t_0 's runtime approximation $\mathcal{R}(t_0)$ with local bound $|\text{Pol}(\ell_1)|$

$\Rightarrow$ set $\mathcal{R}(t_2) := \mathcal{R}(t_0) \cdot |\text{Pol}(\ell_1)|$

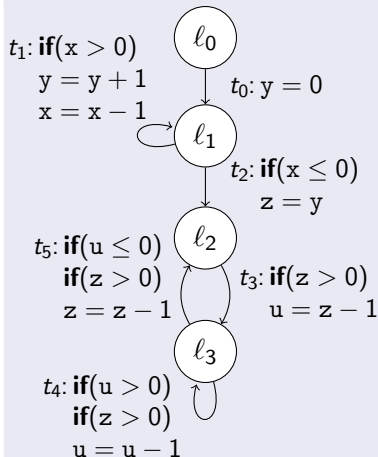
Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\text{Pol}(\ell_1) = 1$, $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = 0$
- $t_2 \in \mathcal{T}'_{\prec}$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_1 , t_2 is used at most $|\text{Pol}(\ell_1)| = 1$ times.
- For global result:
consider how often $\mathcal{T}'$ is reached (by t_0)

$\Rightarrow$ multiply t_0 's runtime approximation $\mathcal{R}(t_0)$ with local bound $|\text{Pol}(\ell_1)|$

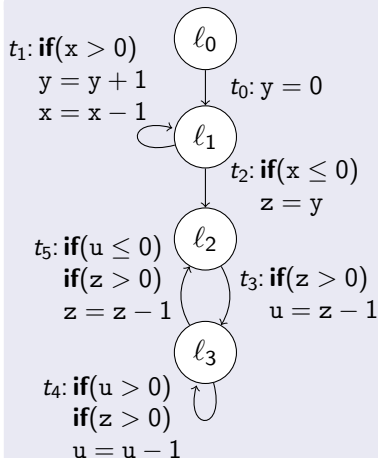
$\Rightarrow$ set $\mathcal{R}(t_2) := \mathcal{R}(t_0) \cdot |\text{Pol}(\ell_1)| = 1 \cdot 1 = 1$



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$, $\mathcal{R}(t_2) = 1$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v
- Novel modular use of PRFs just for **subset** $\mathcal{T}' = \{t_1, \dots, t_5\}$
- $\text{Pol}(\ell_1) = 1$, $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = 0$
- $t_2 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_1 , t_2 is used at most $|\text{Pol}(\ell_1)| = 1$ times.
- For global result:
consider how often $\mathcal{T}'$ is reached (by t_0)



$\Rightarrow$ multiply t_0 's runtime approximation $\mathcal{R}(t_0)$ with local bound $|\text{Pol}(\ell_1)|$

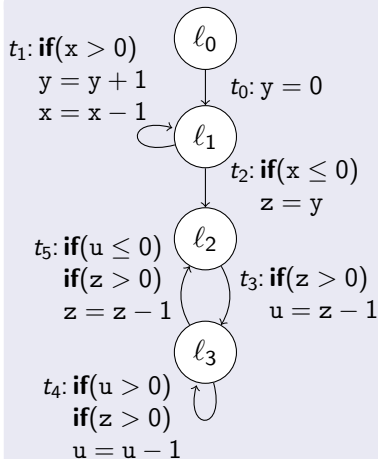
$\Rightarrow$ set $\mathcal{R}(t_2) := \mathcal{R}(t_0) \cdot |\text{Pol}(\ell_1)| = 1 \cdot 1 = 1$

Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$, $\mathcal{R}(t_2) = 1$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

$\mathcal{R}(t_5)$ depends on **size** of z after transition t_2



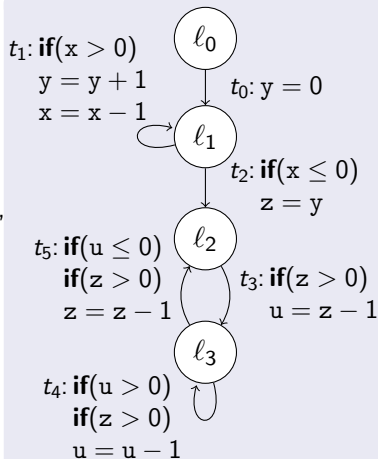
Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$, $\mathcal{R}(t_2) = 1$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

$\mathcal{R}(t_5)$ depends on **size** of z after transition t_2 , i.e.,

$\mathcal{R}(t_5)$ depends on $\mathcal{S}(t_2, z')$



Modular Runtime Bounds from Polynomial Rank Functions

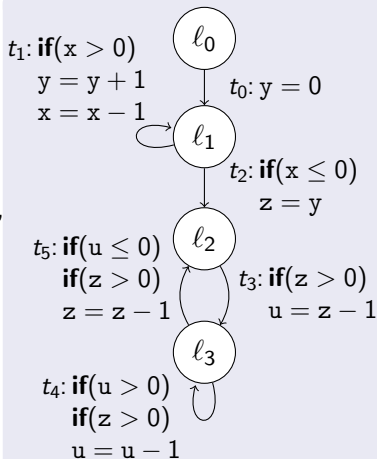
Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$, $\mathcal{R}(t_2) = 1$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

$\mathcal{R}(t_5)$ depends on **size** of z after transition t_2 , i.e.,

$\mathcal{R}(t_5)$ depends on $\mathcal{S}(t_2, z')$

$\Rightarrow$ use **size bounds** to compute **runtime bounds**



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

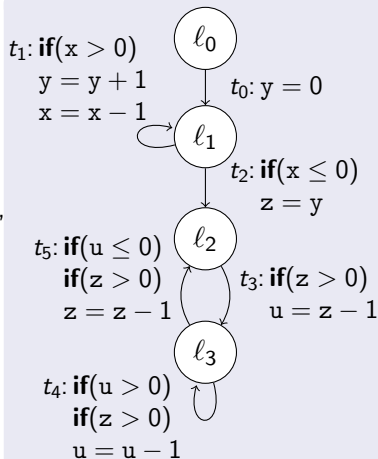
- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$, $\mathcal{R}(t_2) = 1$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

$\mathcal{R}(t_5)$ depends on **size** of z after transition t_2 , i.e.,

$\mathcal{R}(t_5)$ depends on $\mathcal{S}(t_2, z')$

⇒ use **size bounds** to compute **runtime bounds**

⇒ use **runtime bounds** to compute **size bounds**



Modular Runtime Bounds from Polynomial Rank Functions

Current approximations

- $\mathcal{R}(t_0) = 1$, $\mathcal{R}(t_1) = |x|$, $\mathcal{R}(t_2) = 1$
- $\mathcal{R}(t) = ?$ for all other transitions t
- $\mathcal{S}(t, v') = ?$ for all t and variables v

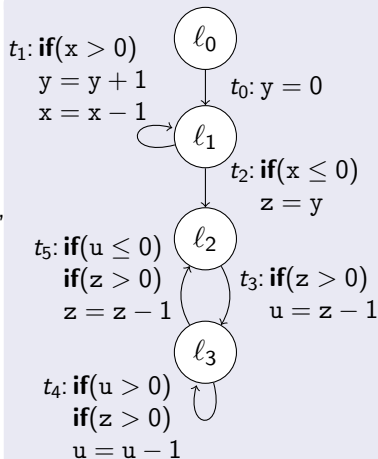
$\mathcal{R}(t_5)$ depends on **size** of z after transition t_2 , i.e.,

$\mathcal{R}(t_5)$ depends on $\mathcal{S}(t_2, z')$

⇒ use **size bounds** to compute **runtime bounds**

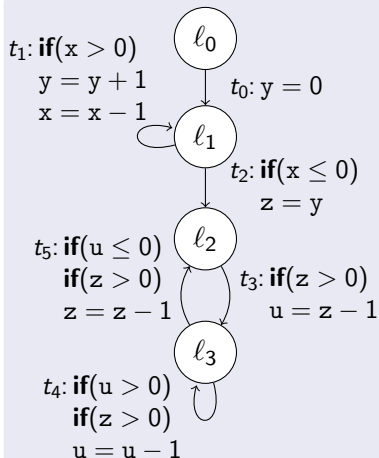
⇒ use **runtime bounds** to compute **size bounds**

⇒ alternate between computing **runtime** and **size** bounds



Size Bounds

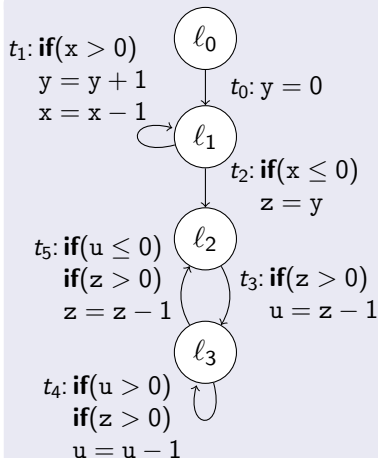
Result Variable Graph (RVG)



Size Bounds

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)

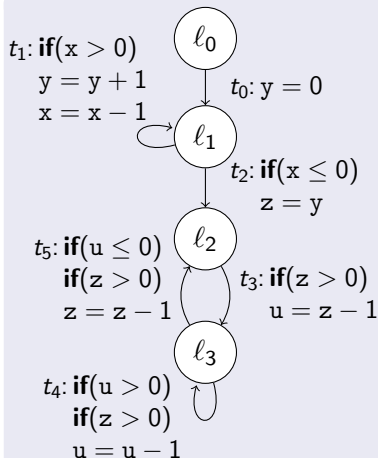


Size Bounds

$ t_0, x' $	$ t_0, y' $	$ t_0, z' $	$ t_0, u' $
$ t_1, x' $	$ t_1, y' $	$ t_1, z' $	$ t_1, u' $
$ t_2, x' $	$ t_2, y' $	$ t_2, z' $	$ t_2, u' $
$ t_3, x' $	$ t_3, y' $	$ t_3, z' $	$ t_3, u' $
$ t_4, x' $	$ t_4, y' $	$ t_4, z' $	$ t_4, u' $
$ t_5, x' $	$ t_5, y' $	$ t_5, z' $	$ t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)

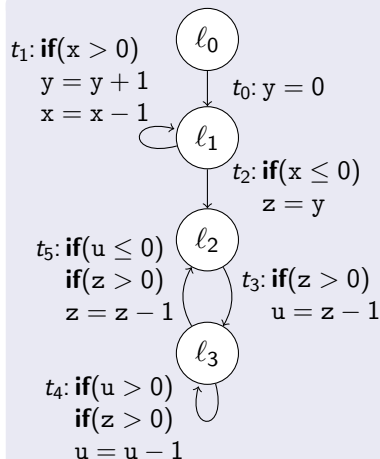


Size Bounds

$ t_0, x' $	$ t_0, y' $	$ t_0, z' $	$ t_0, u' $
$ t_1, x' $	$ t_1, y' $	$ t_1, z' $	$ t_1, u' $
$ t_2, x' $	$ t_2, y' $	$ t_2, z' $	$ t_2, u' $
$ t_3, x' $	$ t_3, y' $	$ t_3, z' $	$ t_3, u' $
$ t_4, x' $	$ t_4, y' $	$ t_4, z' $	$ t_4, u' $
$ t_5, x' $	$ t_5, y' $	$ t_5, z' $	$ t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- Local size approximation** $\mathcal{S}_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'

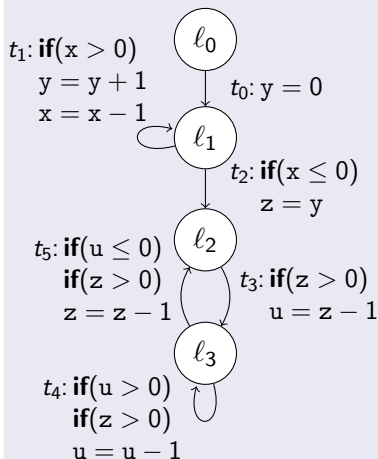


Size Bounds

$ t_0, x' $	$ t_0, y' $	$ t_0, z' $	$ t_0, u' $
$ t_1, x' $	$ y + 1 \geq t_1, y' $	$ t_1, z' $	$ t_1, u' $
$ t_2, x' $	$ t_2, y' $	$ t_2, z' $	$ t_2, u' $
$ t_3, x' $	$ t_3, y' $	$ t_3, z' $	$ t_3, u' $
$ t_4, x' $	$ t_4, y' $	$ t_4, z' $	$ t_4, u' $
$ t_5, x' $	$ t_5, y' $	$ t_5, z' $	$ t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- Local size approximation** $\mathcal{S}_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'

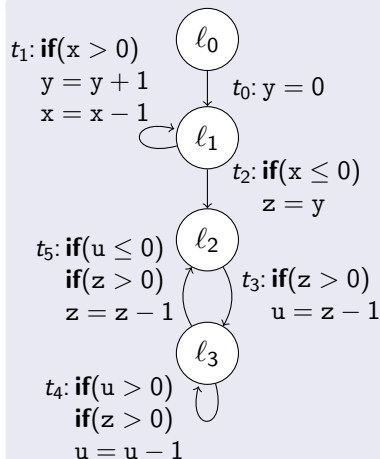


Size Bounds

$ t_0, x' $	$ t_0, y' $	$ t_0, z' $	$ t_0, u' $
$ x \geq t_1, x' $	$ y + 1 \geq t_1, y' $	$ t_1, z' $	$ t_1, u' $
$ t_2, x' $	$ t_2, y' $	$ t_2, z' $	$ t_2, u' $
$ t_3, x' $	$ t_3, y' $	$ t_3, z' $	$ t_3, u' $
$ t_4, x' $	$ t_4, y' $	$ t_4, z' $	$ t_4, u' $
$ t_5, x' $	$ t_5, y' $	$ t_5, z' $	$ t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- Local size approximation** $\mathcal{S}_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'

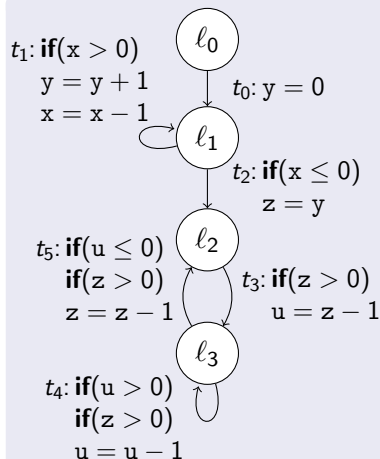


Size Bounds

$ t_0, x' $	$ t_0, y' $	$ t_0, z' $	$ t_0, u' $
$ x \geq t_1, x' $	$ y + 1 \geq t_1, y' $	$ z \geq t_1, z' $	$ u \geq t_1, u' $
$ t_2, x' $	$ t_2, y' $	$ t_2, z' $	$ t_2, u' $
$ t_3, x' $	$ t_3, y' $	$ t_3, z' $	$ t_3, u' $
$ t_4, x' $	$ t_4, y' $	$ t_4, z' $	$ t_4, u' $
$ t_5, x' $	$ t_5, y' $	$ t_5, z' $	$ t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- Local size approximation** $\mathcal{S}_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'



Size Bounds

$$|t_0, x'| \quad 0 \geq |t_0, y'| \quad |t_0, z'| \quad |t_0, u'|$$

$$|x| \geq |t_1, x'| \quad |y| + 1 \geq |t_1, y'| \quad |z| \geq |t_1, z'| \quad |u| \geq |t_1, u'|$$

$$|t_2, x'| \quad |t_2, y'| \quad |t_2, z'| \quad |t_2, u'|$$

$$|t_3, x'| \quad |t_3, y'| \quad |t_3, z'| \quad |t_3, u'|$$

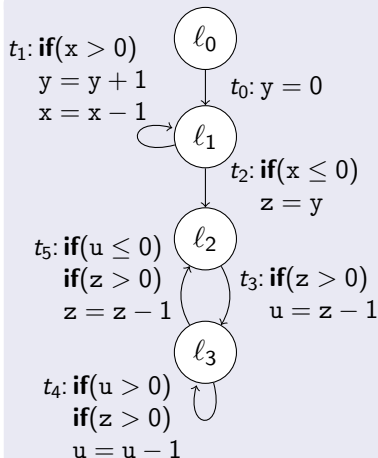
$$|t_4, x'| \quad |t_4, y'| \quad |t_4, z'| \quad |t_4, u'|$$

$$|t_5, x'| \quad |t_5, y'| \quad |t_5, z'| \quad |t_5, u'|$$

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)

- Local size approximation** $\mathcal{S}_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'



Size Bounds

$$|x| \geq |t_0, x'| \quad 0 \geq |t_0, y'| \quad |z| \geq |t_0, z'| \quad |u| \geq |t_0, u'|$$

$$|x| \geq |t_1, x'| \quad |y| + 1 \geq |t_1, y'| \quad |z| \geq |t_1, z'| \quad |u| \geq |t_1, u'|$$

$$|t_2, x'| \quad |t_2, y'| \quad |t_2, z'| \quad |t_2, u'|$$

$$|t_3, x'| \quad |t_3, y'| \quad |t_3, z'| \quad |t_3, u'|$$

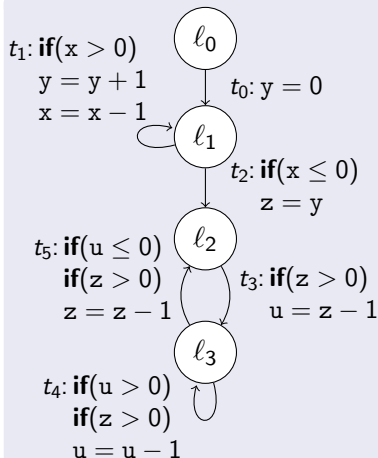
$$|t_4, x'| \quad |t_4, y'| \quad |t_4, z'| \quad |t_4, u'|$$

$$|t_5, x'| \quad |t_5, y'| \quad |t_5, z'| \quad |t_5, u'|$$

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)

- Local size approximation** $\mathcal{S}_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'



Size Bounds

$$|x| \geq |t_0, x'| \quad 0 \geq |t_0, y'| \quad |z| \geq |t_0, z'| \quad |u| \geq |t_0, u'|$$

$$|x| \geq |t_1, x'| \quad |y| + 1 \geq |t_1, y'| \quad |z| \geq |t_1, z'| \quad |u| \geq |t_1, u'|$$

$$|x| \geq |t_2, x'| \quad |y| \geq |t_2, y'| \quad |y| \geq |t_2, z'| \quad |u| \geq |t_2, u'|$$

$$|x| \geq |t_3, x'| \quad |y| \geq |t_3, y'| \quad |z| \geq |t_3, z'| \quad |z| \geq |t_3, u'|$$

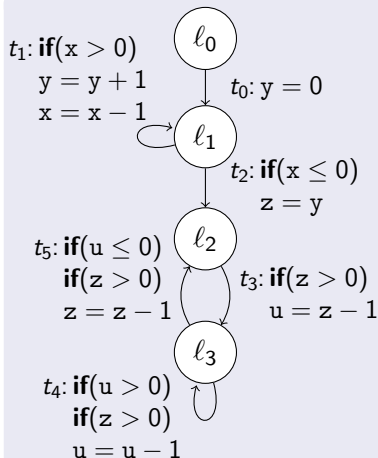
$$|x| \geq |t_4, x'| \quad |y| \geq |t_4, y'| \quad |z| \geq |t_4, z'| \quad |u| \geq |t_4, u'|$$

$$|x| \geq |t_5, x'| \quad |y| \geq |t_5, y'| \quad |z| \geq |t_5, z'| \quad |u| \geq |t_5, u'|$$

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)

- Local size approximation** $\mathcal{S}_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'



Size Bounds

$$|x| \geq |t_0, x'| \quad 0 \geq |t_0, y'| \quad |z| \geq |t_0, z'| \quad |u| \geq |t_0, u'|$$

$$|x| \geq |t_1, x'| \quad |y| + 1 \geq |t_1, y'| \quad |z| \geq |t_1, z'| \quad |u| \geq |t_1, u'|$$

$$|x| \geq |t_2, x'| \quad |y| \geq |t_2, y'| \quad |y| \geq |t_2, z'| \quad |u| \geq |t_2, u'|$$

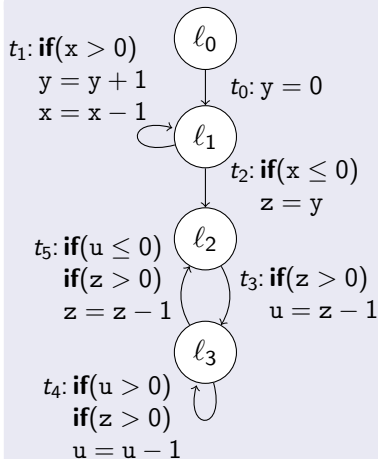
$$|x| \geq |t_3, x'| \quad |y| \geq |t_3, y'| \quad |z| \geq |t_3, z'| \quad |z| \geq |t_3, u'|$$

$$|x| \geq |t_4, x'| \quad |y| \geq |t_4, y'| \quad |z| \geq |t_4, z'| \quad |u| \geq |t_4, u'|$$

$$|x| \geq |t_5, x'| \quad |y| \geq |t_5, y'| \quad |z| \geq |t_5, z'| \quad |u| \geq |t_5, u'|$$

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- edge from $|\tilde{t}, \tilde{v}'|$ to $|t, v'|$ iff
 - transition $\tilde{t}$ is predecessor of t
 - $\tilde{v}$ occurs in local size bound $\mathcal{S}_l(t, v')$
- Local size approximation $\mathcal{S}_l(t, v')$:**
function in t 's pre-variables which bounds t 's post-variable v'

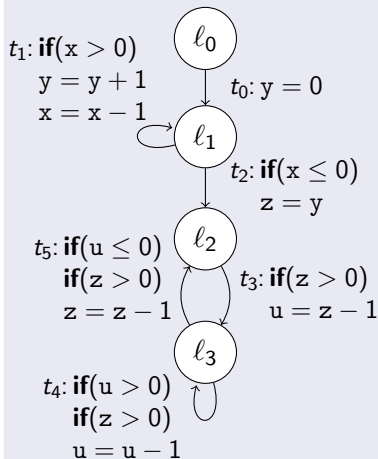


Size Bounds

$ x \geq t_0, x' $	$0 \geq t_0, y' $	$ z \geq t_0, z' $	$ u \geq t_0, u' $
$ x \geq t_1, x' $	$ y + 1 \geq t_1, y' $	$ z \geq t_1, z' $	$ u \geq t_1, u' $
$ x \geq t_2, x' $	$ y \geq t_2, y' $	$ y \geq t_2, z' $	$ u \geq t_2, u' $
$ x \geq t_3, x' $	$ y \geq t_3, y' $	$ z \geq t_3, z' $	$ z \geq t_3, u' $
$ x \geq t_4, x' $	$ y \geq t_4, y' $	$ z \geq t_4, z' $	$ u \geq t_4, u' $
$ x \geq t_5, x' $	$ y \geq t_5, y' $	$ z \geq t_5, z' $	$ u \geq t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- edge from $|\tilde{t}, \tilde{v}'|$ to $|t, v'|$ iff
 - transition $\tilde{t}$ is predecessor of t
 - $\tilde{v}$ occurs in local size bound $\mathcal{S}_l(t, v')$
- Local size approximation $\mathcal{S}_l(t, v')$:**
function in t 's pre-variables which bounds t 's post-variable v'

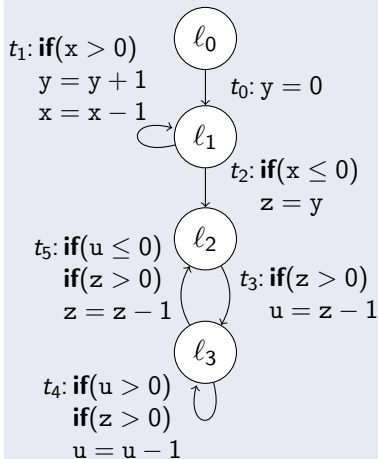


Size Bounds

$ x \geq t_0, x' $	$0 \geq t_0, y' $	$ z \geq t_0, z' $	$ u \geq t_0, u' $
$\downarrow \curvearrowright$	$\downarrow \curvearrowright$	$\downarrow \curvearrowright$	$\downarrow \curvearrowright$
$ x \geq t_1, x' $	$ y + 1 \geq t_1, y' $	$ z \geq t_1, z' $	$ u \geq t_1, u' $
$ x \geq t_2, x' $	$ y \geq t_2, y' $	$ y \geq t_2, z' $	$ u \geq t_2, u' $
$ x \geq t_3, x' $	$ y \geq t_3, y' $	$ z \geq t_3, z' $	$ z \geq t_3, u' $
$ x \geq t_4, x' $	$ y \geq t_4, y' $	$ z \geq t_4, z' $	$ u \geq t_4, u' $
$ x \geq t_5, x' $	$ y \geq t_5, y' $	$ z \geq t_5, z' $	$ u \geq t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- edge from $|\tilde{t}, \tilde{v}'|$ to $|t, v'|$ iff
 - transition $\tilde{t}$ is predecessor of t
 - $\tilde{v}$ occurs in local size bound $\mathcal{S}_l(t, v')$
- Local size approximation $\mathcal{S}_l(t, v')$:**
function in t 's pre-variables which bounds t 's post-variable v'

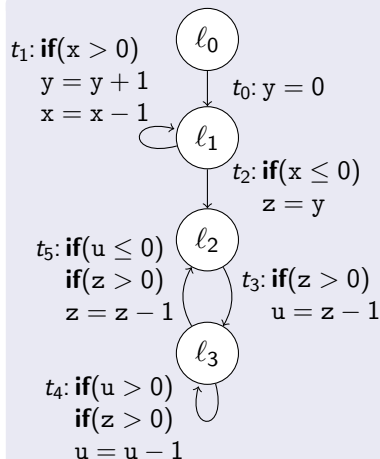


Size Bounds

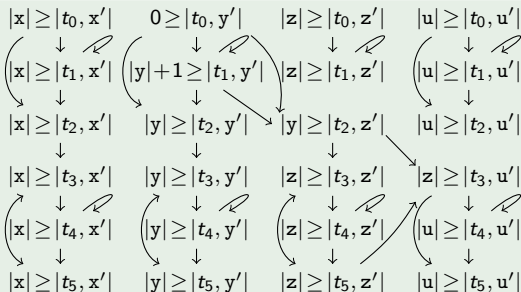
$ x \geq t_0, x' $	$0 \geq t_0, y' $	$ z \geq t_0, z' $	$ u \geq t_0, u' $
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$ x \geq t_1, x' $	$ y + 1 \geq t_1, y' $	$ z \geq t_1, z' $	$ u \geq t_1, u' $
$\downarrow$			
$ x \geq t_2, x' $	$ y \geq t_2, y' $	$ y \geq t_2, z' $	$ u \geq t_2, u' $
$\downarrow$			
$ x \geq t_3, x' $	$ y \geq t_3, y' $	$ z \geq t_3, z' $	$ z \geq t_3, u' $
$\downarrow$			
$ x \geq t_4, x' $	$ y \geq t_4, y' $	$ z \geq t_4, z' $	$ u \geq t_4, u' $
$\downarrow$			
$ x \geq t_5, x' $	$ y \geq t_5, y' $	$ z \geq t_5, z' $	$ u \geq t_5, u' $

Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- edge from $|\tilde{t}, \tilde{v}'|$ to $|t, v'|$ iff
 - transition $\tilde{t}$ is predecessor of t
 - $\tilde{v}$ occurs in local size bound $\mathcal{S}_l(t, v')$
- Local size approximation $\mathcal{S}_l(t, v')$:**
function in t 's pre-variables which bounds t 's post-variable v'

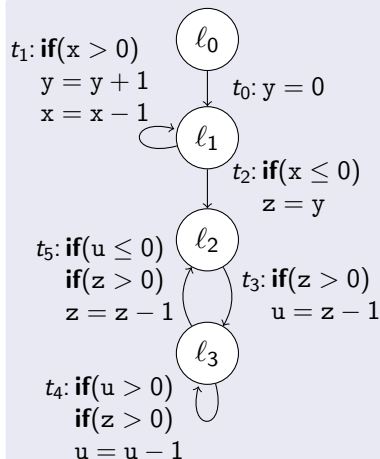


Size Bounds

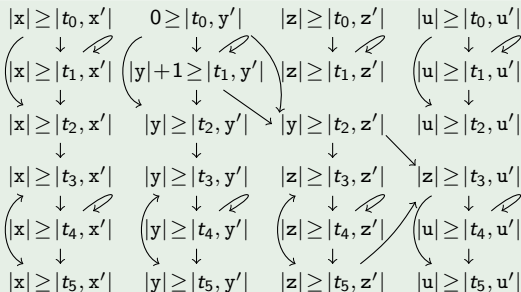


Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- edge from $|\tilde{t}, \tilde{v}'|$ to $|t, v'|$ iff
 - transition $\tilde{t}$ is predecessor of t
 - $\tilde{v}$ occurs in local size bound $\mathcal{S}_l(t, v')$
- Local size approximation $\mathcal{S}_l(t, v')$:**
function in t 's pre-variables which bounds t 's post-variable v'

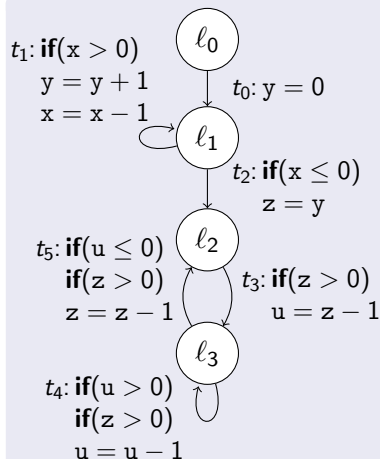


Size Bounds

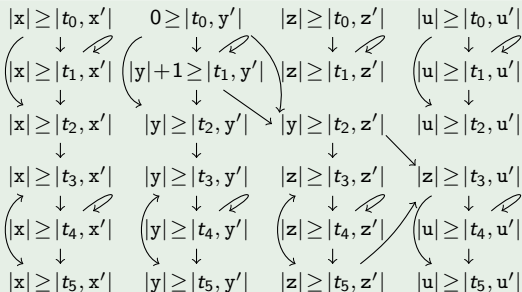


Result Variable Graph (RVG)

- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- edge from $|\tilde{t}, \tilde{v}'|$ to $|t, v'|$ iff
 - transition $\tilde{t}$ is predecessor of t
 - $\tilde{v}$ occurs in local size bound $S_l(t, v')$
- Local size approximation** $S_l(t, v')$:
function in t 's pre-variables which bounds t 's post-variable v'
- Problem:** $S_l(t, v')$ approximates size of v' w.r.t. t 's pre-variables



Size Bounds



Result Variable Graph (RVG)

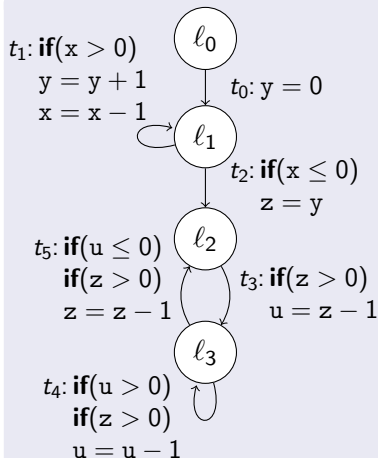
- nodes are **result variables** $|t, v'|$
(denotes absolute value of v after transition t)
- edge from $|\tilde{t}, \tilde{v}'|$ to $|t, v'|$ iff
 - transition $\tilde{t}$ is predecessor of t and
 - $\tilde{v}$ occurs in local size bound $S_I(t, v')$

Local size approximation $S_I(t, v')$:

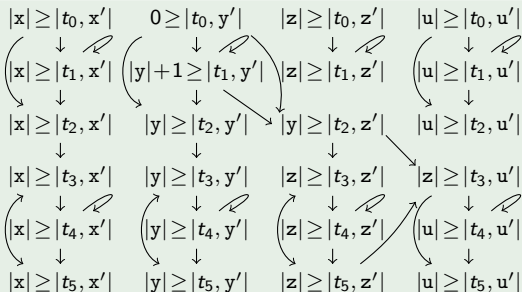
function in t 's pre-variables which bounds t 's post-variable v'

Problem: $S_I(t, v')$ approximates size of v' w.r.t. t 's pre-variables

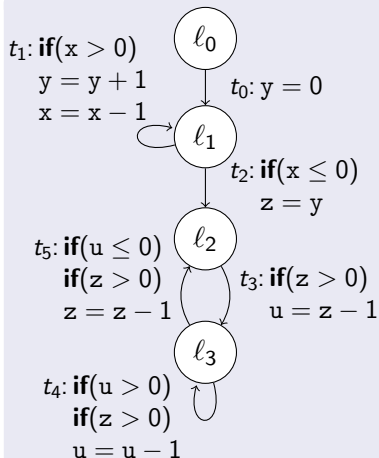
Goal: $S(t, v')$ which approximates v' w.r.t. initial variable values at program start



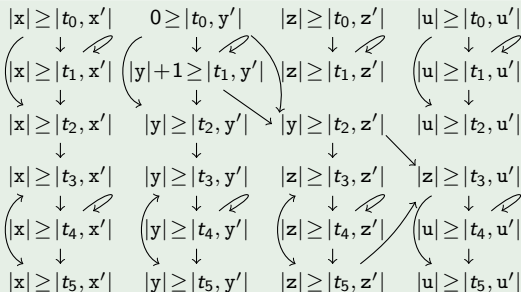
Procedure SizeBounds



Procedure SizeBounds

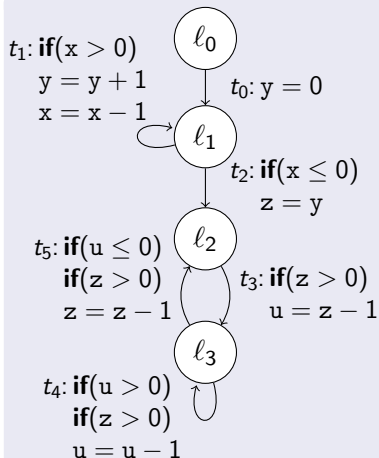


Procedure SizeBounds

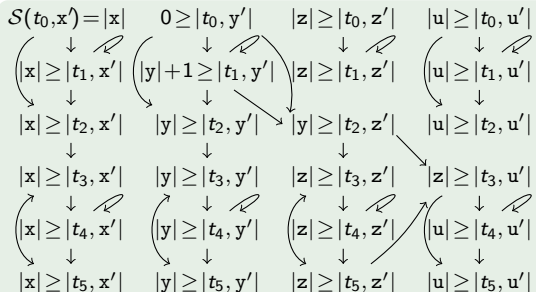


Procedure SizeBounds

- For initial transitions t , set $\mathcal{S}(t, v') := S_I(t, v')$

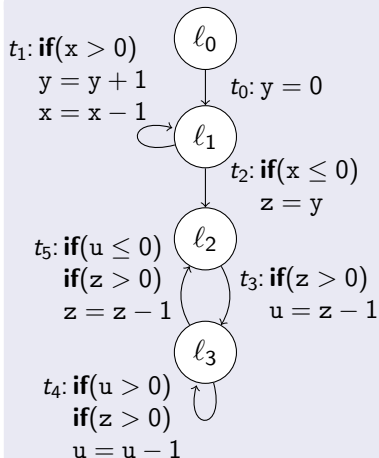


Procedure SizeBounds

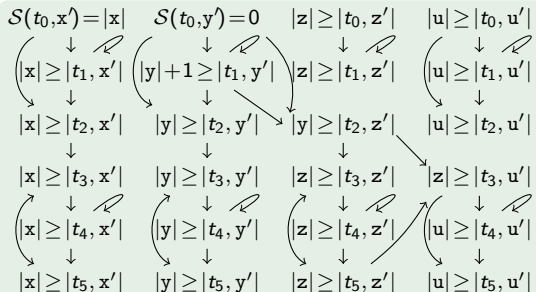


Procedure SizeBounds

- For initial transitions t , set $\mathcal{S}(t, v') := S_I(t, v')$

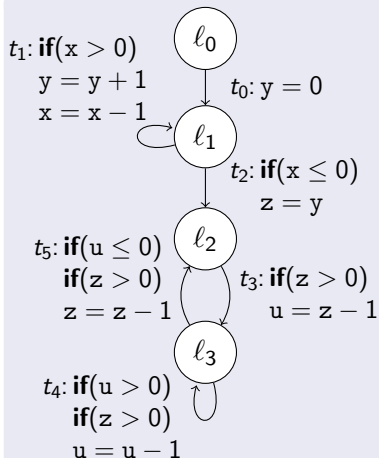


Procedure SizeBounds

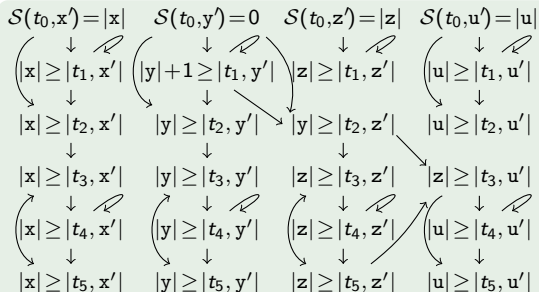


Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$

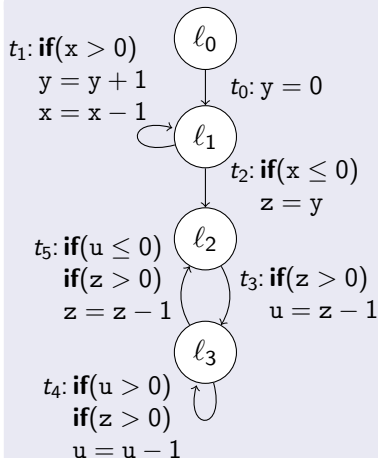


Procedure SizeBounds

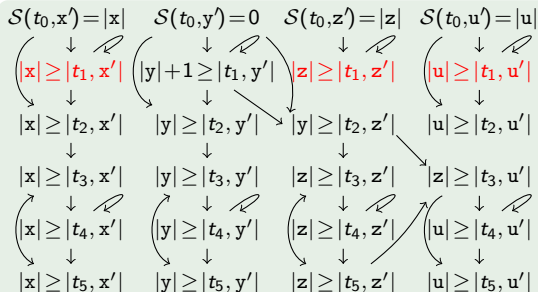


Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$

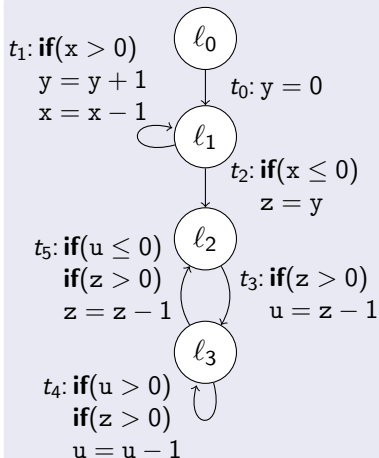


Procedure SizeBounds



Procedure SizeBounds

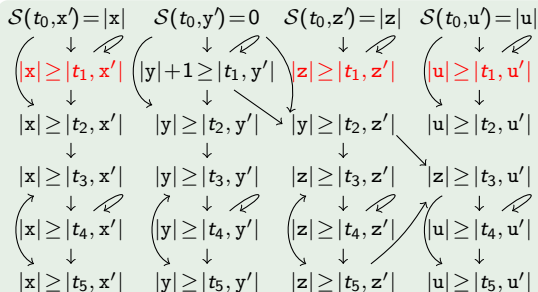
- For initial transitions t , set $S(t, v') := S_I(t, v')$



For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$

Procedure SizeBounds

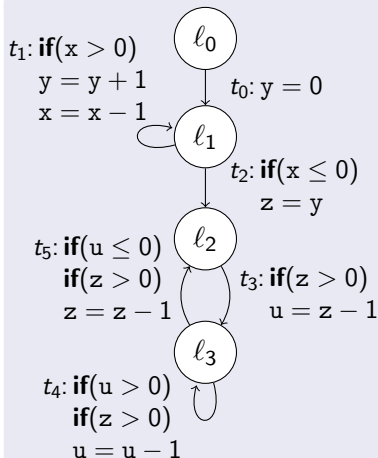


Procedure SizeBounds

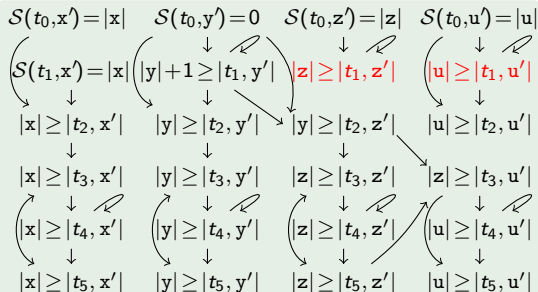
- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\}$

For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$



Procedure SizeBounds

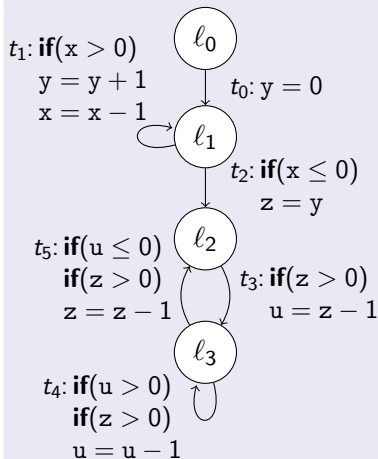


Procedure SizeBounds

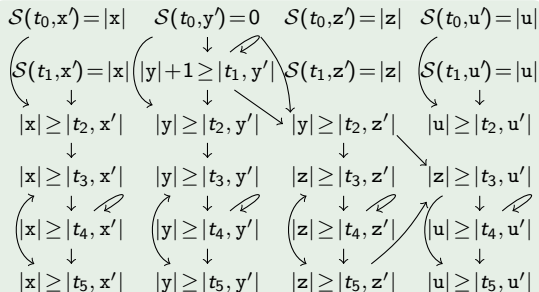
- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\}$

For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$



Procedure SizeBounds

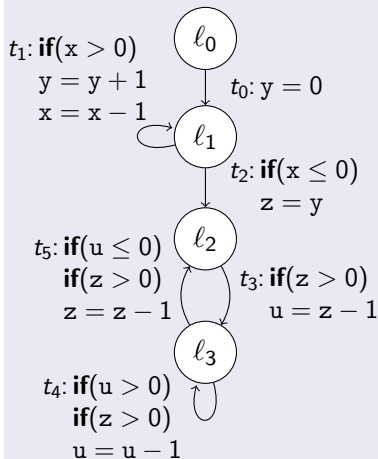


Procedure SizeBounds

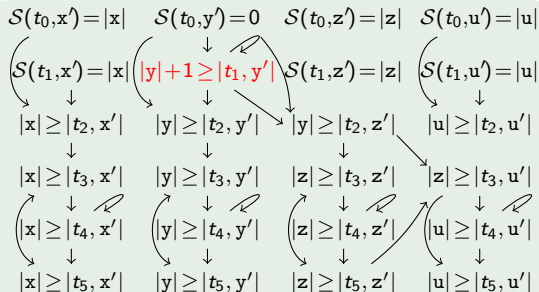
- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\}$

For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$



Procedure SizeBounds

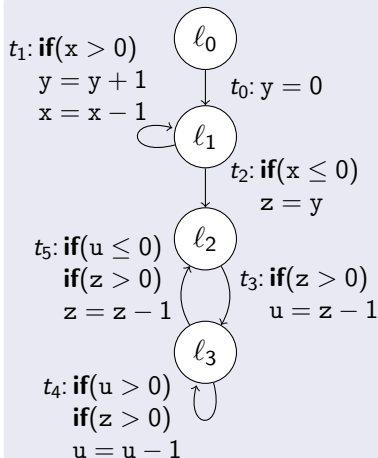


Procedure SizeBounds

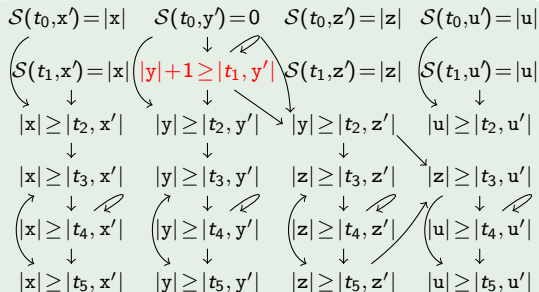
- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\}$

For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$
- $\alpha \in \dagger$ iff $S_I(\alpha) \leq e_\alpha + \max\{|v_1|, \dots, |v_n|\}$ for $e_\alpha \in \mathbb{N}$



Procedure SizeBounds

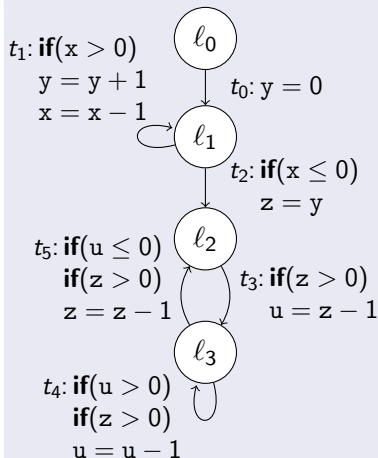


Procedure SizeBounds

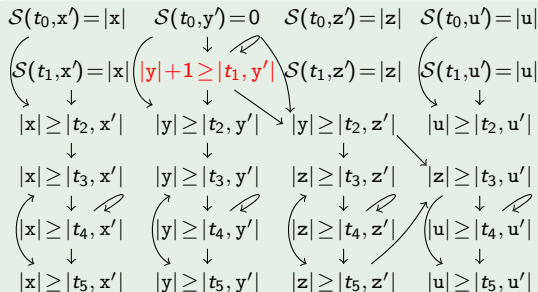
- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$
- $\alpha \in \dagger$ iff $S_I(\alpha) \leq e_\alpha + \max\{|v_1|, \dots, |v_n|\}$ for $e_\alpha \in \mathbb{N}$



Procedure SizeBounds

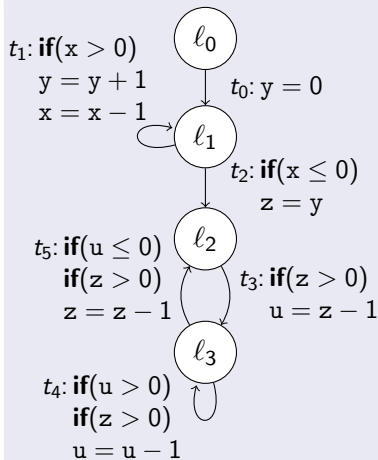


Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

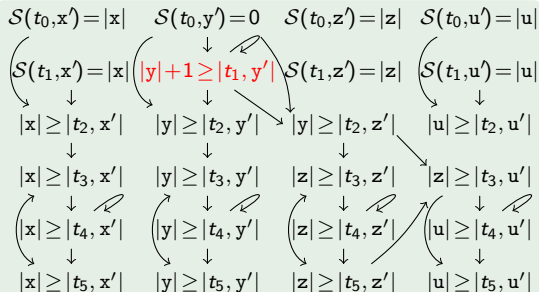
For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$
- $\alpha \in \dagger$ iff $S_I(\alpha) \leq e_\alpha + \max\{|v_1|, \dots, |v_n|\}$ for $e_\alpha \in \mathbb{N}$



$$S(t_1, y') = S(t_0, y') + \mathcal{R}(t_1) \cdot e_{|t_1, y'|}$$

Procedure SizeBounds

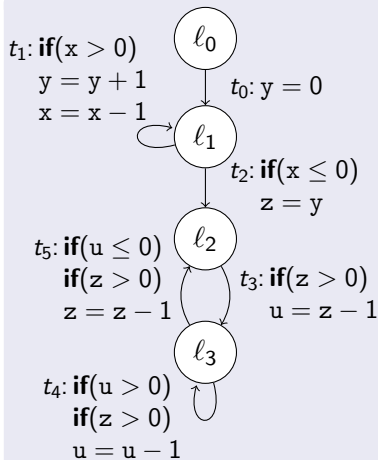


Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

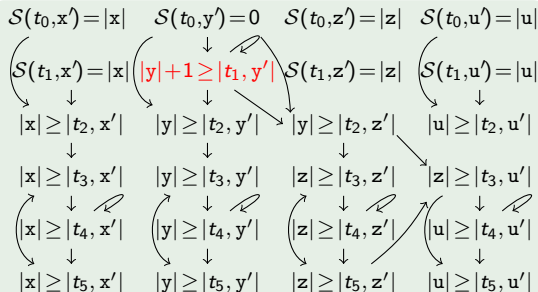
For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$
- $\alpha \in \dagger$ iff $S_I(\alpha) \leq e_\alpha + \max\{|v_1|, \dots, |v_n|\}$ for $e_\alpha \in \mathbb{N}$



$$\begin{aligned}
 S(t_1, y') &= S(t_0, y') + \mathcal{R}(t_1) \cdot e_{|t_1, y'|} \\
 &= 0 + |x| \cdot 1
 \end{aligned}$$

Procedure SizeBounds

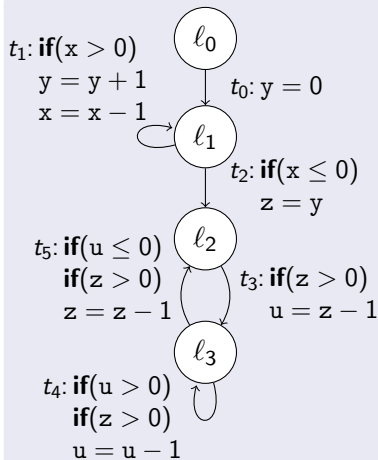


Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

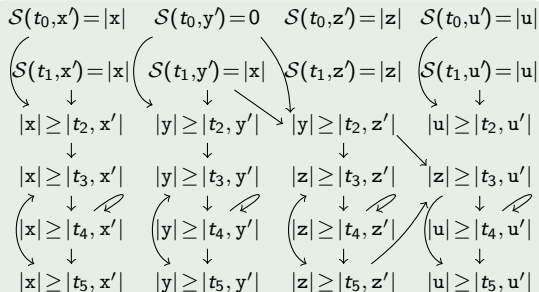
For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$
- $\alpha \in \dagger$ iff $S_I(\alpha) \leq e_\alpha + \max\{|v_1|, \dots, |v_n|\}$ for $e_\alpha \in \mathbb{N}$



$$\begin{aligned}
 S(t_1, y') &= S(t_0, y') + \mathcal{R}(t_1) \cdot e_{|t_1, y'|} \\
 &= 0 + |x| \cdot 1 \\
 &= |x|
 \end{aligned}$$

Procedure SizeBounds

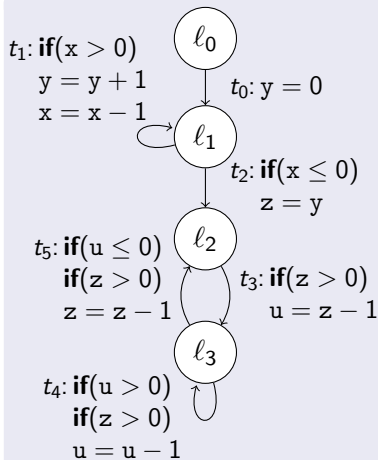


Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

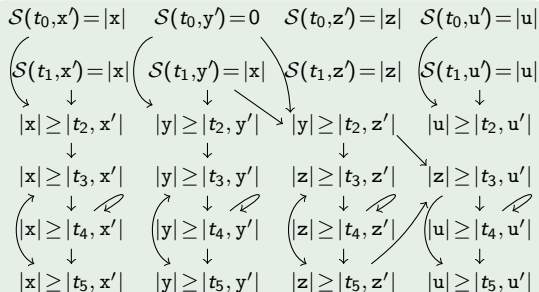
For each result variable α in non-trivial SCCs:

- $\alpha \in \doteq$ iff $S_I(\alpha) \leq \max\{|v_1|, \dots, |v_n|\}$
- $\alpha \in \dagger$ iff $S_I(\alpha) \leq e_\alpha + \max\{|v_1|, \dots, |v_n|\}$ for $e_\alpha \in \mathbb{N}$



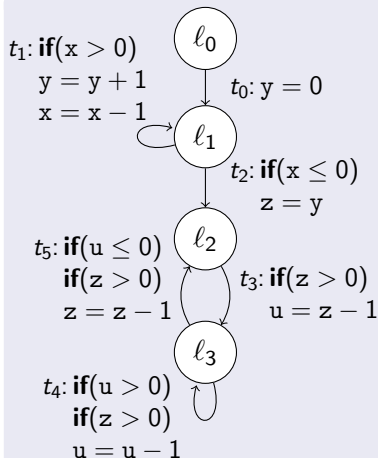
$$\begin{aligned}
 S(t_1, y') &= S(t_0, y') + \mathcal{R}(t_1) \cdot e_{|t_1, y'|} \\
 &= 0 + |x| \cdot 1 \\
 &= |x|
 \end{aligned}$$

Procedure SizeBounds

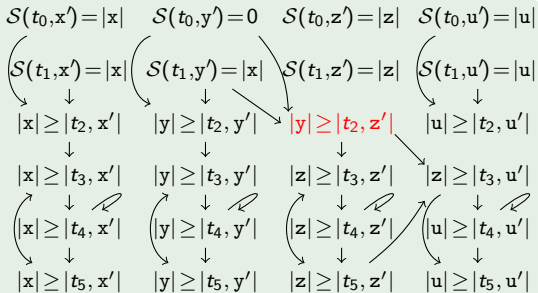


Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $S(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in C} \mathcal{R}(\alpha) \cdot e_\alpha$



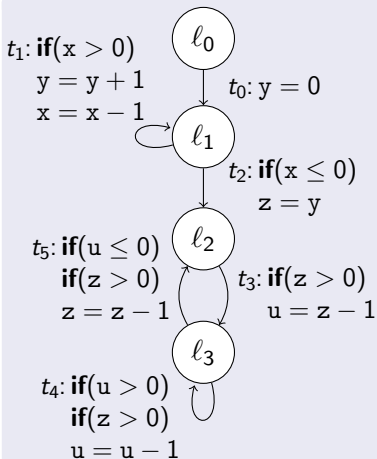
Procedure SizeBounds



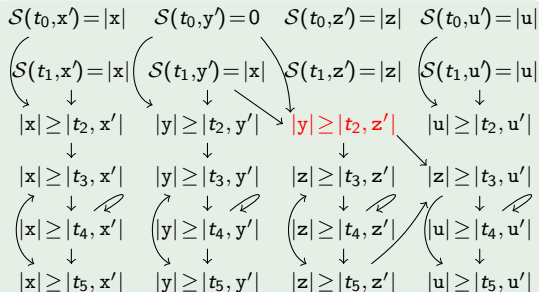
Procedure SizeBounds

- For initial transitions t , set $\mathcal{S}(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $\mathcal{S}(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $\mathcal{S}(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \pm} \mathcal{R}(\alpha) \cdot e_\alpha$

$$\mathcal{S}(t_2, z') = \max \{ \mathcal{S}_I(t_2, z') (\mathcal{S}(t_0, x'), \dots, \mathcal{S}(t_0, u')), \mathcal{S}_I(t_2, z') (\mathcal{S}(t_1, x'), \dots, \mathcal{S}(t_1, u')) \}$$



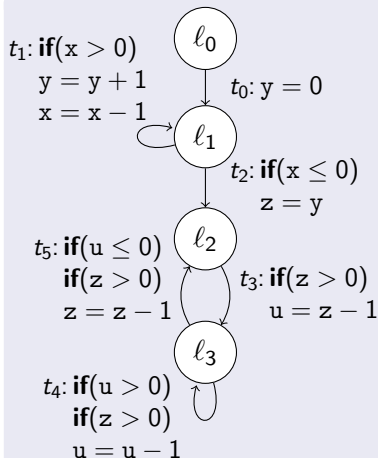
Procedure SizeBounds



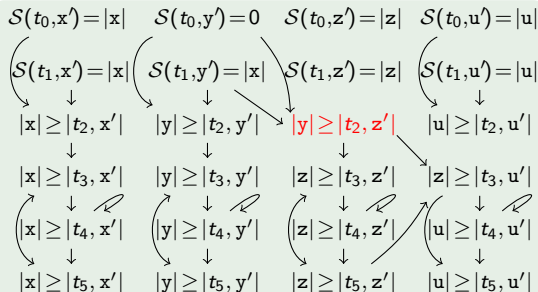
Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $S(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

$$\begin{aligned}
 S(t_2, z') &= \max \{ S_I(t_2, z') (S(t_0, x'), \dots, S(t_0, u')), S_I(t_2, z') (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ |y| (S(t_0, x'), \dots, S(t_0, u')), |y| (S(t_1, x'), \dots, S(t_1, u')) \}
 \end{aligned}$$



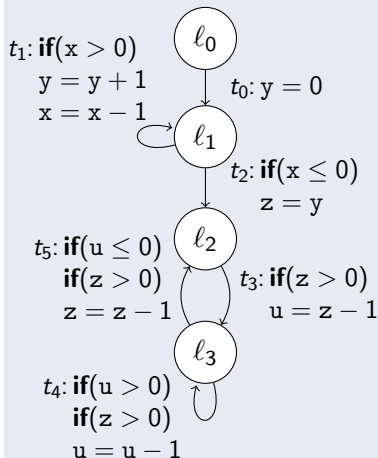
Procedure SizeBounds



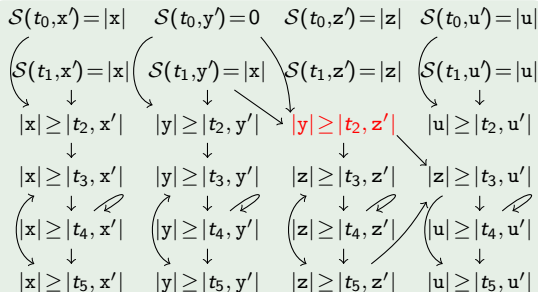
Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $S(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

$$\begin{aligned}
 S(t_2, z') &= \max \{ S_I(t_2, z') (S(t_0, x'), \dots, S(t_0, u')), S_I(t_2, z') (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ |y| (S(t_0, x'), \dots, S(t_0, u')), |y| (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ S(t_0, y'), S(t_1, y') \}
 \end{aligned}$$



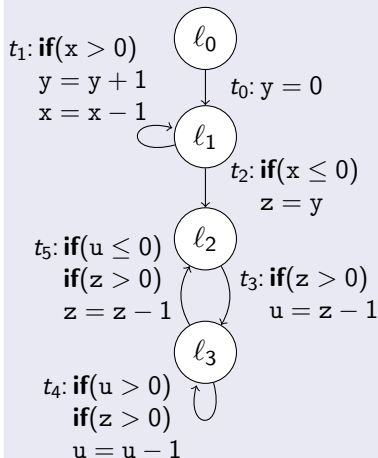
Procedure SizeBounds



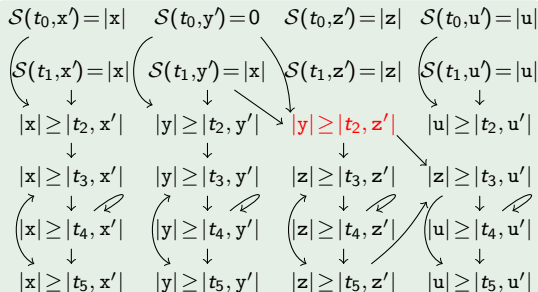
Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $S(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

$$\begin{aligned}
 S(t_2, z') &= \max \{ S_I(t_2, z') (S(t_0, x'), \dots, S(t_0, u')), & S_I(t_2, z') (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ |y| (S(t_0, x'), \dots, S(t_0, u')), & |y| (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ S(t_0, y'), & S(t_1, y') \} \\
 &= \max \{ 0, |x| \}
 \end{aligned}$$



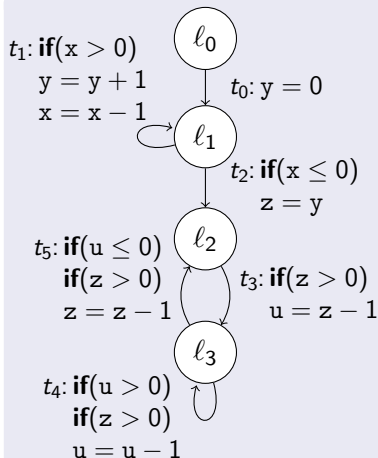
Procedure SizeBounds



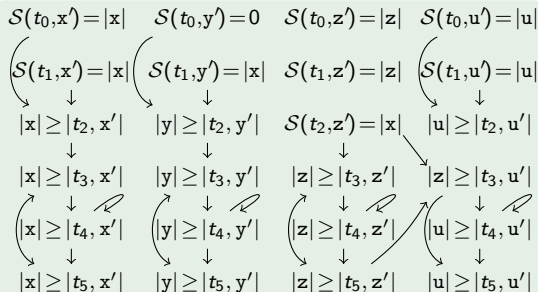
Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $S(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \mathcal{R}} \mathcal{R}(\alpha) \cdot e_\alpha$

$$\begin{aligned}
 S(t_2, z') &= \max \{ S_I(t_2, z') (S(t_0, x'), \dots, S(t_0, u')), S_I(t_2, z') (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ |y| (S(t_0, x'), \dots, S(t_0, u')), |y| (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ S(t_0, y'), S(t_1, y') \} \\
 &= \max \{ 0, |x| \} = |x|
 \end{aligned}$$



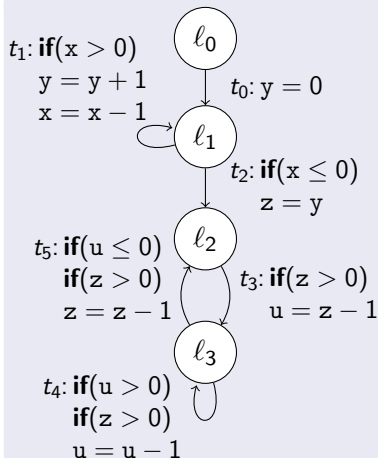
Procedure SizeBounds



Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $S(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \mathcal{R}} \mathcal{R}(\alpha) \cdot e_\alpha$

$$\begin{aligned}
 S(t_2, z') &= \max \{ S_I(t_2, z') (S(t_0, x'), \dots, S(t_0, u')), S_I(t_2, z') (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ |y| (S(t_0, x'), \dots, S(t_0, u')), |y| (S(t_1, x'), \dots, S(t_1, u')) \} \\
 &= \max \{ S(t_0, y'), S(t_1, y') \} \\
 &= \max \{ 0, |x| \} = |x|
 \end{aligned}$$



Procedure SizeBounds

$$S(t_0, x') = |x| \quad S(t_0, y') = 0 \quad S(t_0, z') = |z| \quad S(t_0, u') = |u|$$

$$S(t_1, x') = |x| \quad S(t_1, y') = |x| \quad S(t_1, z') = |z| \quad S(t_1, u') = |u|$$

$$S(t_2, x') = |x| \quad S(t_2, y') = |x| \quad S(t_2, z') = |x| \quad S(t_2, u') = |u|$$

$$S(t_3, x') = |x| \quad S(t_3, y') = |x| \quad S(t_3, z') = |x| \quad S(t_3, u') = |x|$$

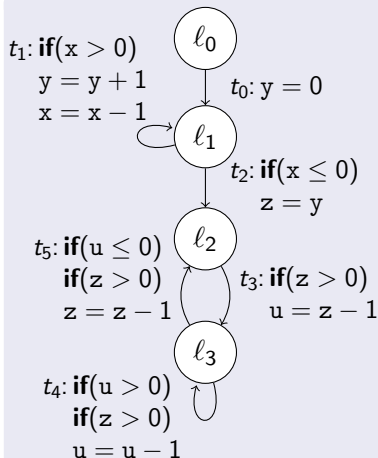
$$S(t_4, x') = |x| \quad S(t_4, y') = |x| \quad S(t_4, z') = |x| \quad S(t_4, u') = |x|$$

$$S(t_5, x') = |x| \quad S(t_5, y') = |x| \quad S(t_5, z') = |x| \quad S(t_5, u') = |x|$$

Procedure SizeBounds

- For initial transitions t , set $S(t, v') := S_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $S(t, v') := \max\{S_I(t, v') (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $S(t, v') := \max\{S(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in C} \mathcal{R}(\alpha) \cdot e_\alpha$

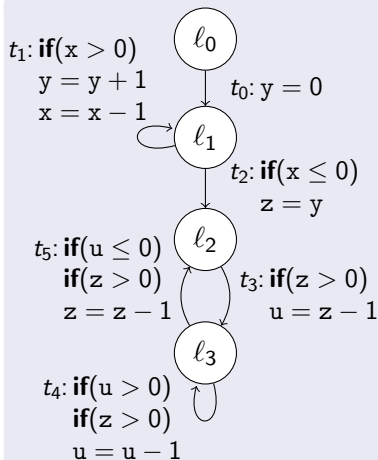
$$\begin{aligned} S(t_2, z') &= \max \{ S_I(t_2, z') (S(t_0, x'), \dots, S(t_0, u')), S_I(t_2, z') (S(t_1, x'), \dots, S(t_1, u')) \} \\ &= \max \{ |y| (S(t_0, x'), \dots, S(t_0, u')), |y| (S(t_1, x'), \dots, S(t_1, u')) \} \\ &= \max \{ S(t_0, y'), S(t_1, y') \} \\ &= \max \{ 0, |x| \} = |x| \end{aligned}$$



Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

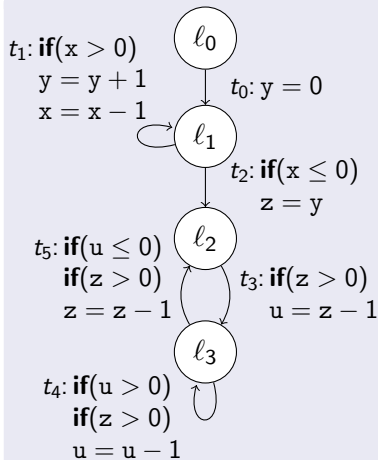


Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$

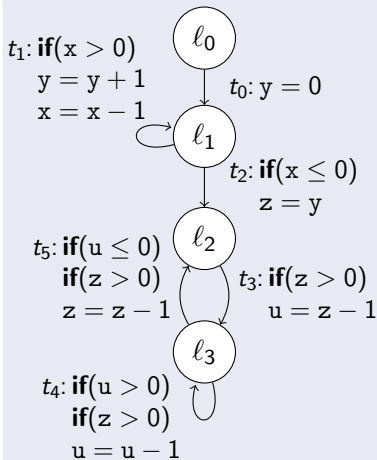


Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\mathcal{P}ol(\ell_2) = \mathcal{P}ol(\ell_3) = z$

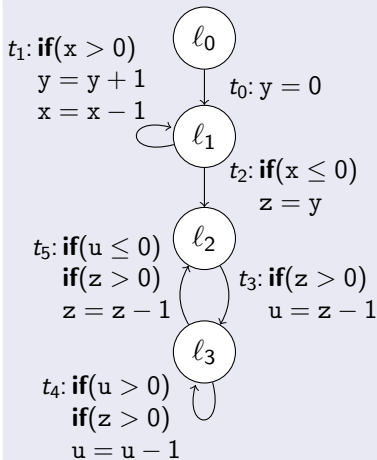


Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\mathcal{P}ol(\ell_2) = \mathcal{P}ol(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'_{\prec}$

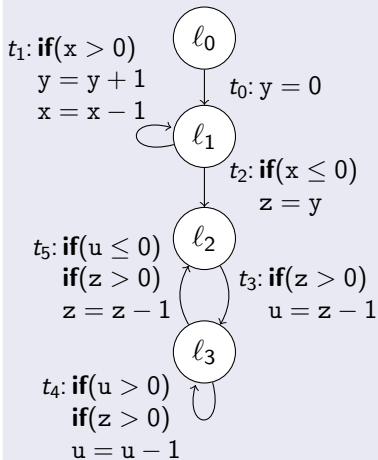


Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\mathcal{Pol}(\ell_2) = \mathcal{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\mathcal{Pol}(\ell_2)| = |z|$ times.

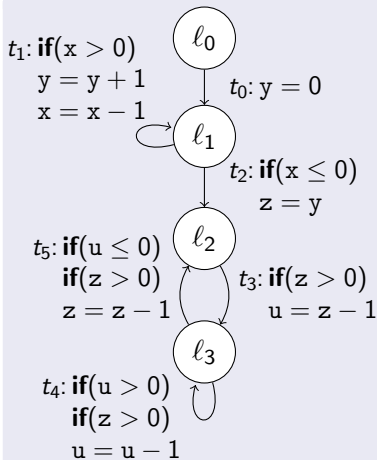


Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\mathcal{Pol}(\ell_2) = \mathcal{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'_{\prec}$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\mathcal{Pol}(\ell_2)| = |z|$ times.
- For global result:

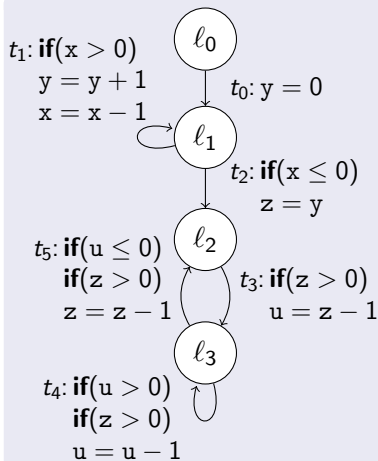


Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\mathcal{Pol}(\ell_2) = \mathcal{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\mathcal{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run

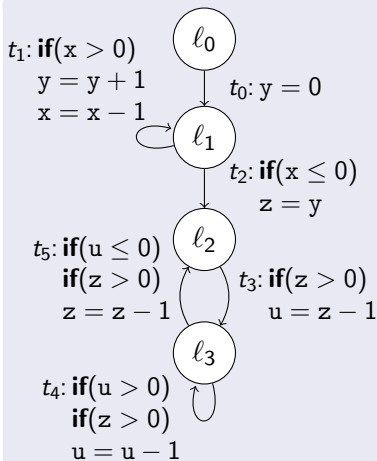


Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run



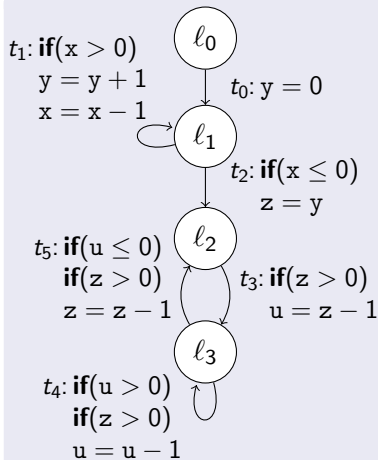
$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (\mathcal{S}(t_2, x'), \dots, \mathcal{S}(t_2, u'))$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



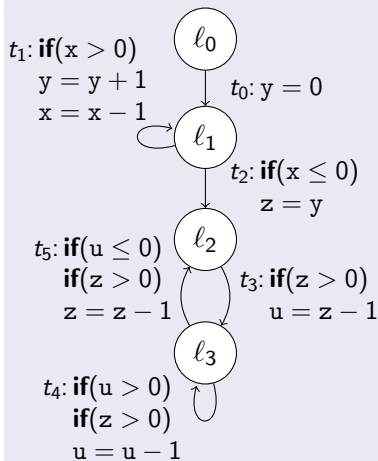
$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (\mathcal{S}(t_2, x'), \dots, \mathcal{S}(t_2, u'))$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable **z** in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

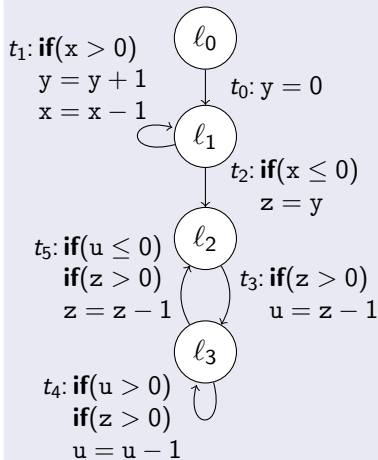
$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

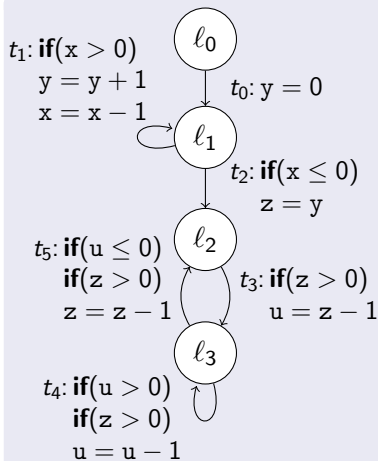
$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable **z** in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

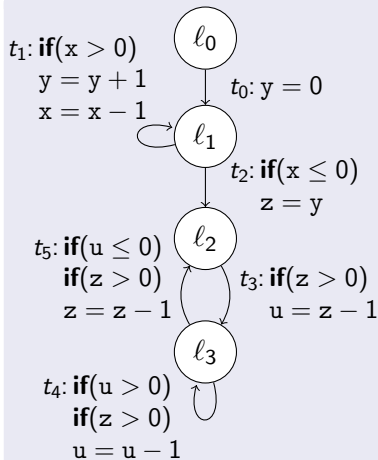
$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot \text{Pol}(\ell_2) (S(t_2, x'), \dots, S(t_2, u'))$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

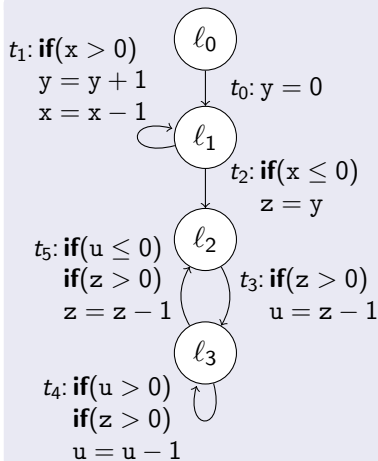
$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot |z| (S(t_2, x'), \dots, S(t_2, u'))$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

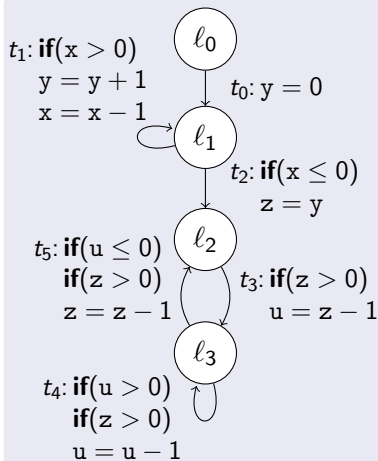
$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot S(t_2, z')$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

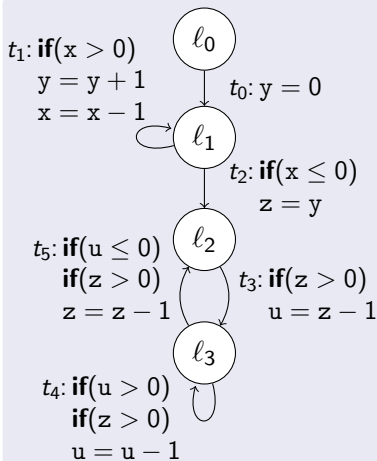
$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot |x|$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

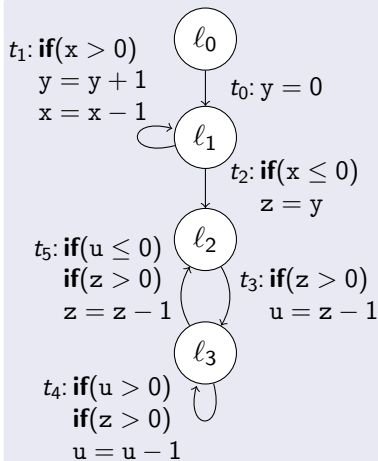
$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot |x| = |x|$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1, \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable z in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot |x| = |x|$

Procedure TimeBounds

Current runtime approximations

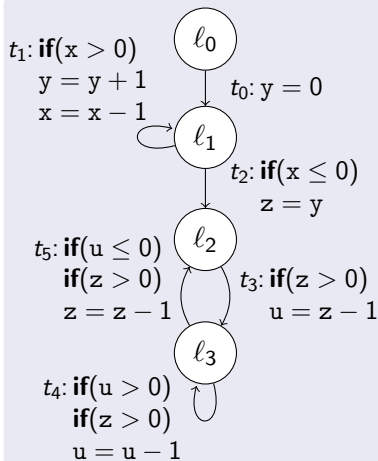
$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1, \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable **z** in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}'} |\text{Pol}(\ell)|$$



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot |x| = |x|$

Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1, \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable **z** in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)

Procedure TimeBounds

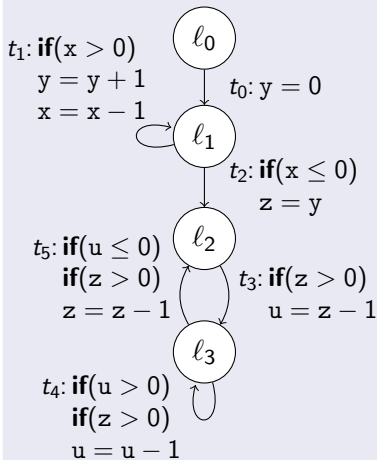
Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$

$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot |x| = |x|$



Procedure TimeBounds

Current runtime approximations

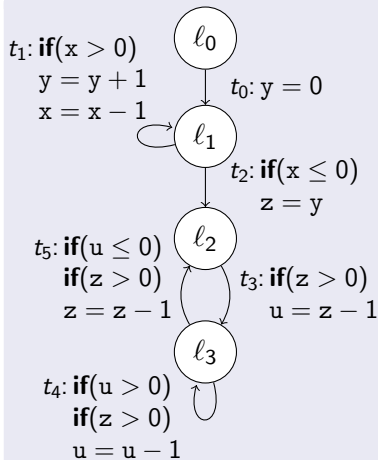
$$\mathcal{R}(t_0) = 1, \mathcal{R}(t_1) = |x|, \mathcal{R}(t_2) = 1, \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4, t_5\}$
- $\text{Pol}(\ell_2) = \text{Pol}(\ell_3) = z$ Thus: $t_5 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_2 , t_5 is used at most $|\text{Pol}(\ell_2)| = |z|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable **z** in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ replace $|\text{Pol}(\ell_2)|$ by $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_2)$ with local bound $|\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u'))$

$\Rightarrow \mathcal{R}(t_5) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| (S(t_2, x'), \dots, S(t_2, u')) = 1 \cdot |x| = |x|$

Procedure TimeBounds

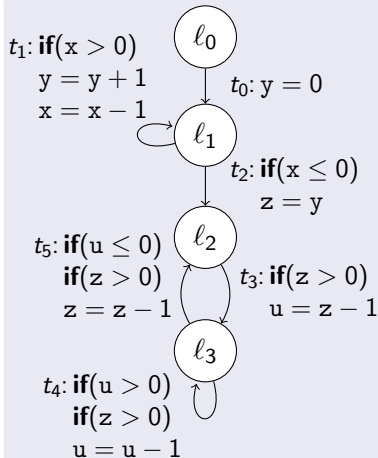
Current runtime approximations

$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}', \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

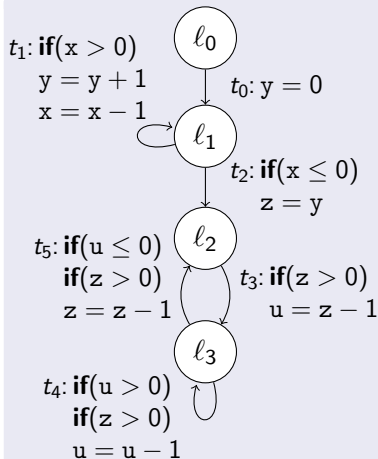
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}', \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

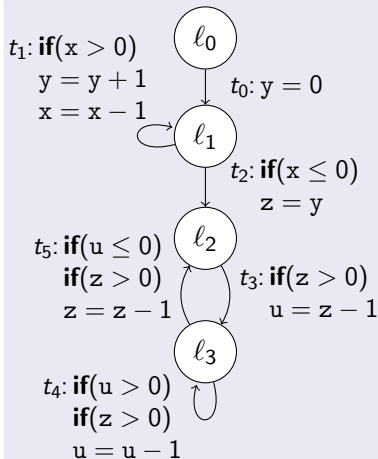
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\mathcal{Pol}(\ell_2) = 1, \mathcal{Pol}(\ell_3) = 0$

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}', \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

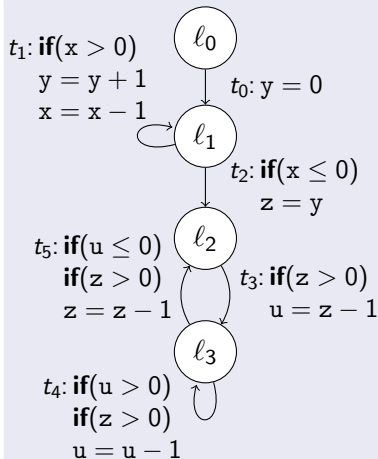
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\mathcal{Pol}(\ell_2) = 1, \mathcal{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}', \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| \cdot (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

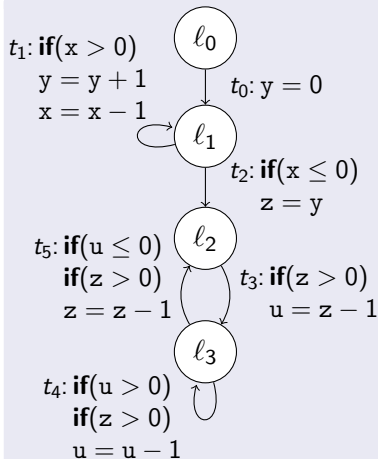
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}', \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

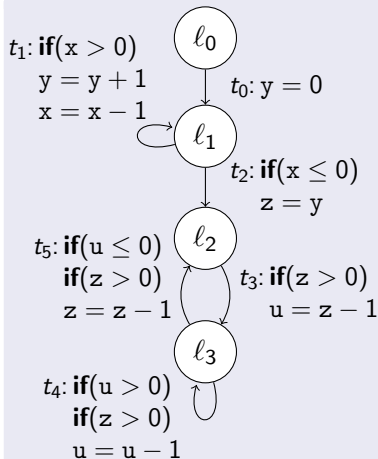
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}', \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

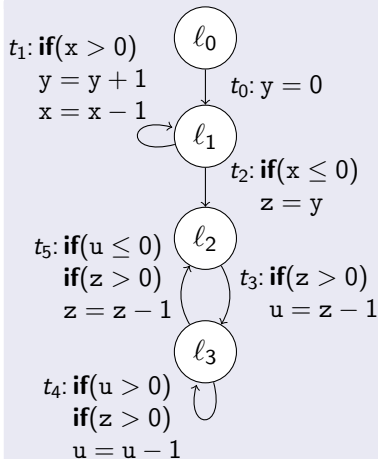
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variables in full run

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\ell \text{ start location in } \mathcal{T}', \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

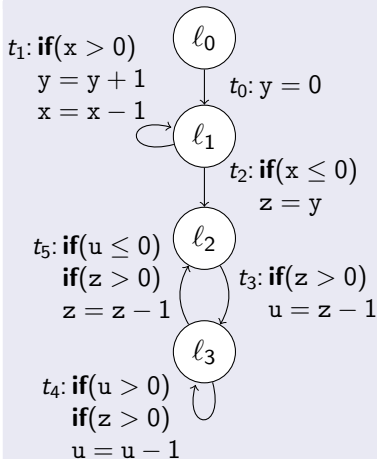
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variables in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2 and t_5)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

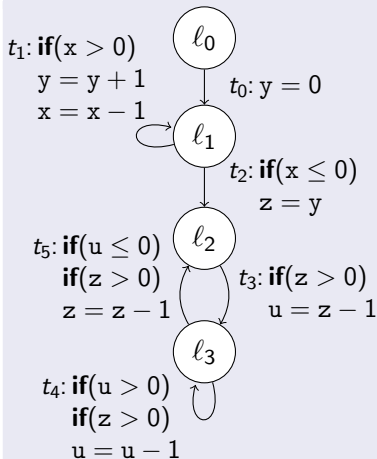
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variables in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2 and t_5)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ multiply runtime approximations $\mathcal{R}(t_2)$ and $\mathcal{R}(t_5)$ with local bound $|\text{Pol}(\ell_2)|$

Procedure TimeBounds

Current runtime approximations

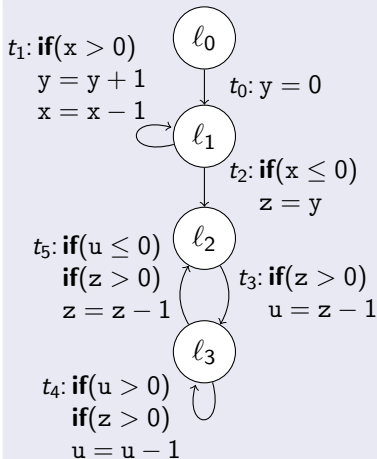
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variables in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2 and t_5)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| \cdot (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ multiply runtime approximations $\mathcal{R}(t_2)$ and $\mathcal{R}(t_5)$ with local bound $|\text{Pol}(\ell_2)|$

$\Rightarrow \mathcal{R}(t_3) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| + \mathcal{R}(t_5) \cdot |\text{Pol}(\ell_2)|$

Procedure TimeBounds

Current runtime approximations

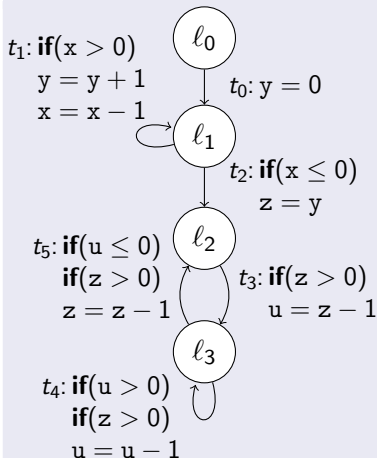
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variables in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2 and t_5)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| \cdot (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ multiply runtime approximations $\mathcal{R}(t_2)$ and $\mathcal{R}(t_5)$ with local bound $|\text{Pol}(\ell_2)|$

$$\Rightarrow \mathcal{R}(t_3) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| + \mathcal{R}(t_5) \cdot |\text{Pol}(\ell_2)| = 1 \cdot 1 + |x| \cdot 1$$

Procedure TimeBounds

Current runtime approximations

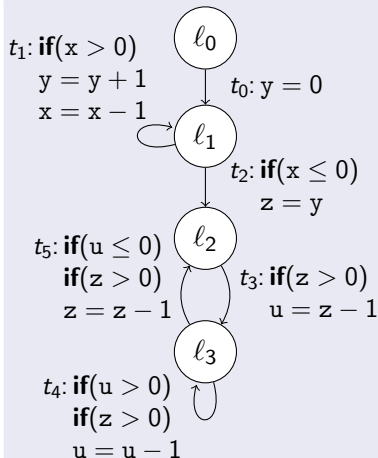
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variables in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2 and t_5)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| \cdot (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ multiply runtime approximations $\mathcal{R}(t_2)$ and $\mathcal{R}(t_5)$ with local bound $|\text{Pol}(\ell_2)|$

$$\Rightarrow \mathcal{R}(t_3) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| + \mathcal{R}(t_5) \cdot |\text{Pol}(\ell_2)| = 1 \cdot 1 + |x| \cdot 1 = 1 + |x|$$

Procedure TimeBounds

Current runtime approximations

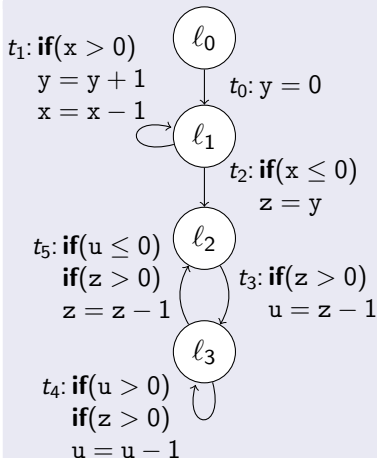
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_3, t_4\}$
- $\text{Pol}(\ell_2) = 1, \text{Pol}(\ell_3) = 0$ Thus: $t_3 \in \mathcal{T}'$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_2 , t_3 is used at most $|\text{Pol}(\ell_2)| = 1$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variables in full run
 - consider how often $\mathcal{T}'$ is reached (by t_2 and t_5)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| \cdot (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ multiply runtime approximations $\mathcal{R}(t_2)$ and $\mathcal{R}(t_5)$ with local bound $|\text{Pol}(\ell_2)|$

$$\Rightarrow \mathcal{R}(t_3) := \mathcal{R}(t_2) \cdot |\text{Pol}(\ell_2)| + \mathcal{R}(t_5) \cdot |\text{Pol}(\ell_2)| = 1 \cdot 1 + |x| \cdot 1 = 1 + |x|$$

Procedure TimeBounds

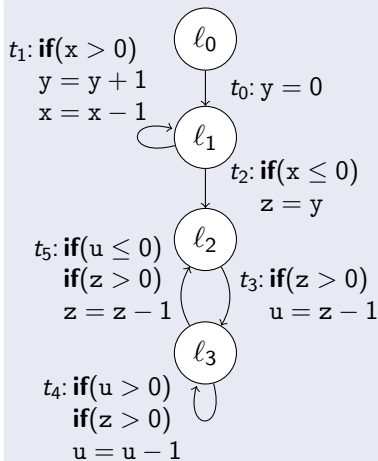
Current runtime approximations

$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

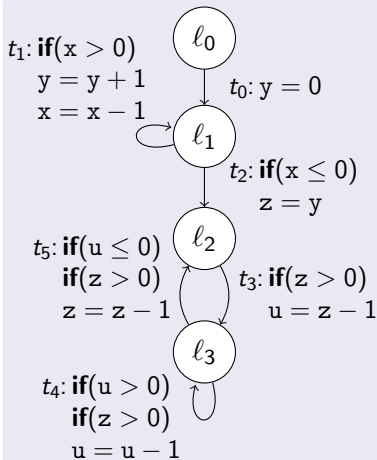
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

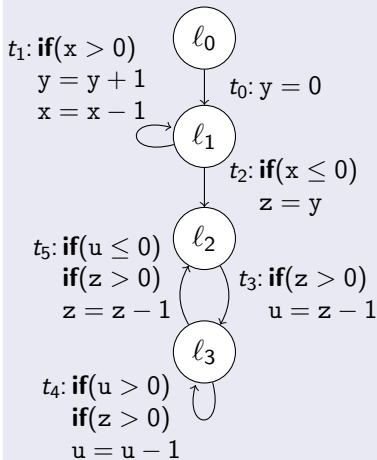
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = u$

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

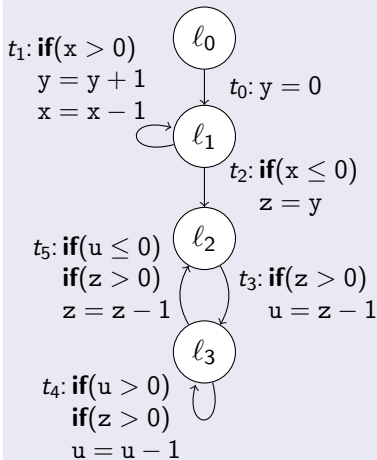
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_\gamma$

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_\gamma$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

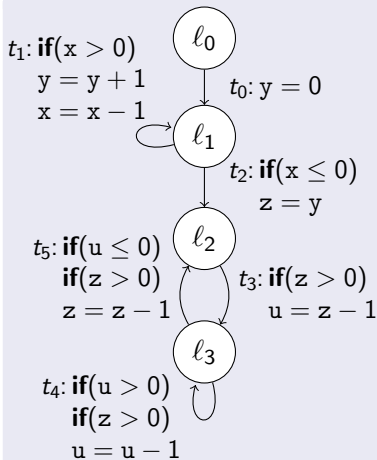
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\mathcal{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_\gamma$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_3 , t_4 is used at most $|\mathcal{Pol}(\ell_3)| = |\mathbf{u}|$ times.

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_\gamma$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| \cdot (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

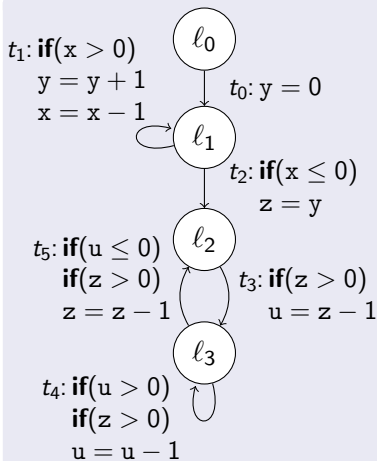
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_\prec$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |\mathbf{u}|$ times.
- For global result:

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_\prec$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| \cdot (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

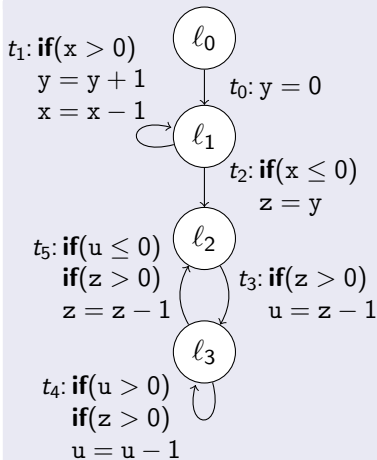
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_\gamma$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |\mathbf{u}|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable $\mathbf{u}$ in full run

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_\gamma$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

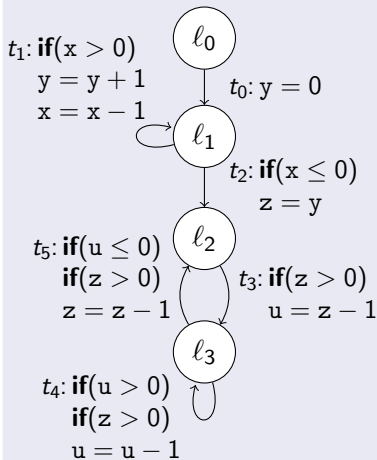
- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_{\prec}$
- In executions **restricted to** $\mathcal{T}'$ starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |\mathbf{u}|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable $\mathbf{u}$ in full run

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_{\prec}$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$

$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, \mathbf{x}'), \dots, S(t_3, \mathbf{u}'))$



Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

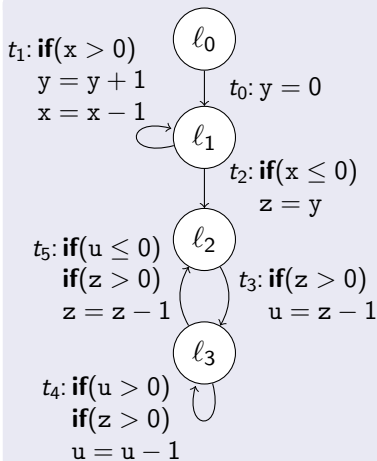
- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |\mathbf{u}|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable $\mathbf{u}$ in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$

$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, \mathbf{x}'), \dots, S(t_3, \mathbf{u}'))$



Procedure TimeBounds

Current runtime approximations

$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = u$ Thus: $t_4 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |u|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable **u** in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

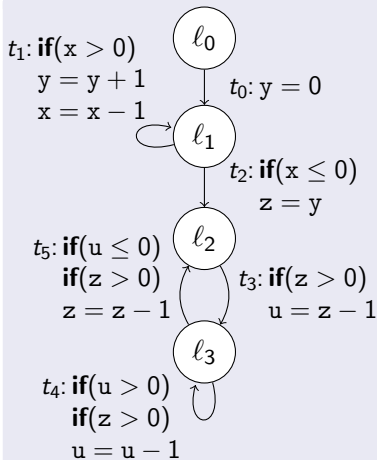
Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$

$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$



Procedure TimeBounds

Current runtime approximations

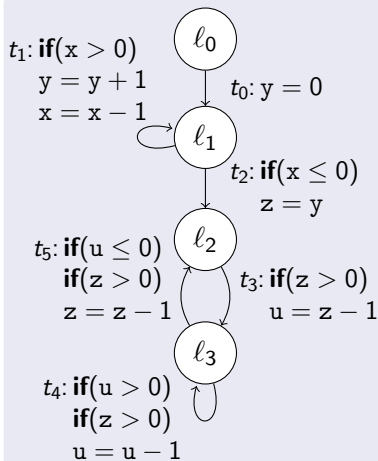
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = u$ Thus: $t_4 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |u|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable **u** in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow \mathcal{R}(t_4) := \mathcal{R}(t_3) \cdot |\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

Procedure TimeBounds

Current runtime approximations

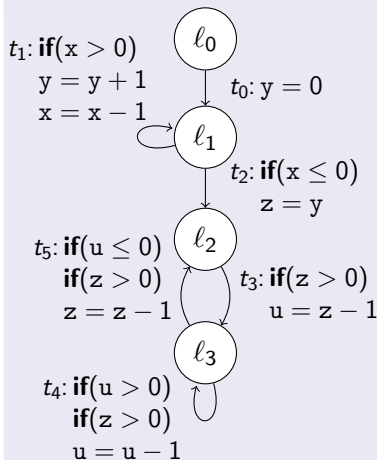
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = u$ Thus: $t_4 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |u|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable u in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow \mathcal{R}(t_4) := \mathcal{R}(t_3) \cdot |\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u')) = (1 + |x|) \cdot \text{Pol}(\ell_3) (S(t_3, x'), \dots, S(t_3, u'))$

Procedure TimeBounds

Current runtime approximations

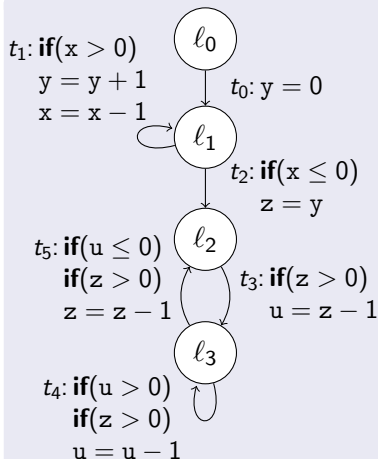
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = u$ Thus: $t_4 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |u|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable u in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow \mathcal{R}(t_4) := \mathcal{R}(t_3) \cdot |\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u')) = (1 + |x|) \cdot |u| (S(t_3, x'), \dots, S(t_3, u'))$

Procedure TimeBounds

Current runtime approximations

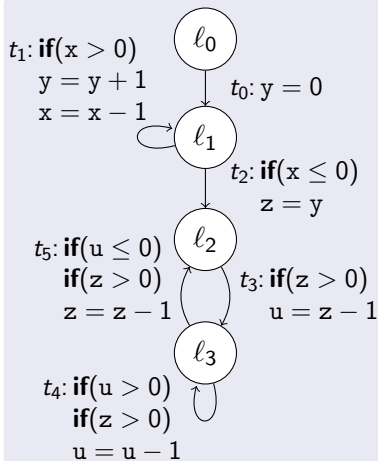
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_{\prec}$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |\mathbf{u}|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable $\mathbf{u}$ in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_{\prec}$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow \mathcal{R}(t_4) := \mathcal{R}(t_3) \cdot |\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u')) = (1 + |x|) \cdot S(t_3, u')$

Procedure TimeBounds

Current runtime approximations

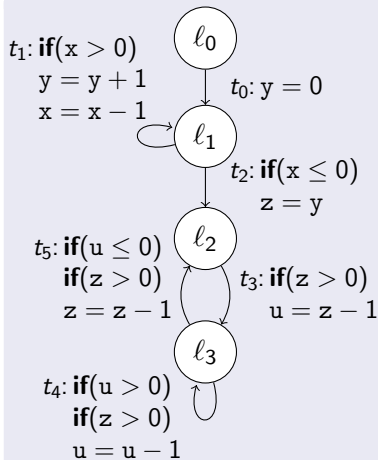
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_{\prec}$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |\mathbf{u}|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable $\mathbf{u}$ in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_{\prec}$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow \mathcal{R}(t_4) := \mathcal{R}(t_3) \cdot |\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u')) = (1 + |x|) \cdot |x|$

Procedure TimeBounds

Current runtime approximations

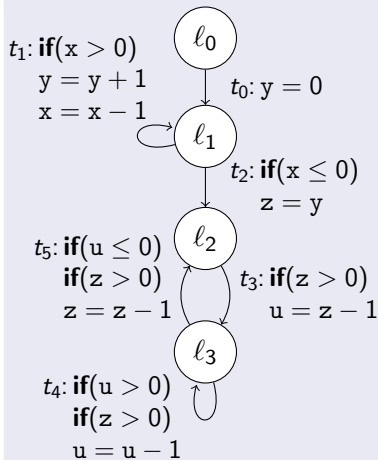
$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = \mathbf{u}$ Thus: $t_4 \in \mathcal{T}'_{\prec}$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |\mathbf{u}|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable $\mathbf{u}$ in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Procedure TimeBounds

Let Pol be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'_{\prec}$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}' \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\text{Pol}(\ell)| (S(\tilde{t}, v'_1), \dots, S(\tilde{t}, v'_n))$$



$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow \mathcal{R}(t_4) := \mathcal{R}(t_3) \cdot |\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u')) = (1 + |x|) \cdot |x| = |x| + x^2$

Procedure TimeBounds

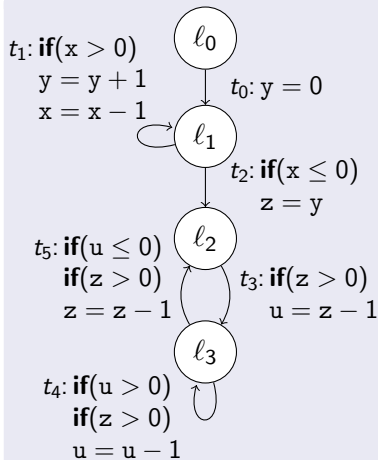
Current runtime approximations

$$\mathcal{R}(t_0) = \mathcal{R}(t_2) = 1, \mathcal{R}(t_1) = \mathcal{R}(t_5) = |x|, \mathcal{R}(t_3) = 1 + |x|$$

- use PRF just for **subset** $\mathcal{T}' = \{t_4\}$
- $\text{Pol}(\ell_3) = u$ Thus: $t_4 \in \mathcal{T}'$
- In executions **restricted to $\mathcal{T}'$** starting in ℓ_3 , t_4 is used at most $|\text{Pol}(\ell_3)| = |u|$ times.
- For global result:
 - consider value of $\mathcal{T}'$'s initial variable u in full run
 - consider how often $\mathcal{T}'$ is reached (by t_3)

Overall runtime approximation

$$\mathcal{R}(t_0) + \dots + \mathcal{R}(t_5) = 3 + 4 \cdot |x| + x^2$$



$\Rightarrow$ replace $|\text{Pol}(\ell_3)|$ by $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow$ multiply runtime approximation $\mathcal{R}(t_3)$ with local bound $|\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u'))$

$\Rightarrow \mathcal{R}(t_4) := \mathcal{R}(t_3) \cdot |\text{Pol}(\ell_3)| (S(t_3, x'), \dots, S(t_3, u')) = (1 + |x|) \cdot |x| = |x| + x^2$

Runtime and Size Complexity of Integer Programs

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}', \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| (\mathcal{S}(\tilde{t}, v'_1), \dots, \mathcal{S}(\tilde{t}, v'_n))$$

Procedure SizeBounds

- For initial transitions t , set $\mathcal{S}(t, v') := \mathcal{S}_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $\mathcal{S}(t, v') := \max\{\mathcal{S}_I(t, v') (\mathcal{S}(\tilde{t}, v'_1), \dots, \mathcal{S}(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $\mathcal{S}(t, v') := \max\{\mathcal{S}(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

Runtime and Size Complexity of Integer Programs

Procedure TimeBounds

Let $\mathcal{Pol}$ be PRF for $\mathcal{T}' \subseteq \mathcal{T}$. For every $t \in \mathcal{T}'$, set

$$\mathcal{R}(t) = \sum_{\substack{\ell \text{ start location in } \mathcal{T}', \\ \tilde{t} \notin \mathcal{T}' \text{ reaches } \ell}} \mathcal{R}(\tilde{t}) \cdot |\mathcal{Pol}(\ell)| \cdot (\mathcal{S}(\tilde{t}, v'_1), \dots, \mathcal{S}(\tilde{t}, v'_n))$$

Procedure SizeBounds

- For initial transitions t , set $\mathcal{S}(t, v') := \mathcal{S}_I(t, v')$
- For other $|t, v'|$ in trivial SCCs, set $\mathcal{S}(t, v') := \max\{\mathcal{S}_I(t, v')(\mathcal{S}(\tilde{t}, v'_1), \dots, \mathcal{S}(\tilde{t}, v'_n)) \mid \tilde{t} \text{ predecessor of } t\}$
- For $|t, v'|$ in non-trivial SCCs C , set $\mathcal{S}(t, v') := \max\{\mathcal{S}(\alpha) \mid \alpha \text{ predecessor of } C\} + \sum_{\alpha \in \dagger} \mathcal{R}(\alpha) \cdot e_\alpha$

Overall Procedure

Input: Program $\mathcal{T}$

$$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$$

while there exist t, v with

$\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$$

$$\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$$

for all SCCs C of the RVG
in topological order **do**

$$\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$$

done

done

Output: Runtime approximation $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
- compute **size bounds** by combining local variable changes with **runtime bounds**

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time
 - linear rank functions usually sufficient

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time
 - linear rank functions usually sufficient
 - approach scales to larger programs

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time
 - linear rank functions usually sufficient
 - approach scales to larger programs
- Extensions

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time
 - linear rank functions usually sufficient
 - approach scales to larger programs
- Extensions
 - recursive procedure calls

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time
 - linear rank functions usually sufficient
 - approach scales to larger programs
- Extensions
 - recursive procedure calls
 - advanced cost measures

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time
 - linear rank functions usually sufficient
 - approach scales to larger programs
- Extensions
 - recursive procedure calls
 - advanced cost measures
 - separate analysis of procedures and loops

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$

Related Work on Symbolic Complexity Analysis

- PUBS/COSTA (Albert et al), Loopus (Zuleger et al)
- Rank (Alias et al)
- SPEED (Gulwani et al)
- RAML (Hofmann et al)
- ABC (Blanc, Kovács et al)
- CASLOG (Debray, Lin), CiaoPP (Hermenegildo et al)
- TcT (Moser et al), AProVE (Giesl et al)
- ...

Related Work and Experiments

Related Work on Symbolic Complexity Analysis

- PUBS/COSTA (Albert et al)
- Rank (Alias et al)

Prototype Implementation KoAT

Related Work and Experiments

Related Work on Symbolic Complexity Analysis

- PUBS/COSTA (Albert et al)
- Rank (Alias et al)

Prototype Implementation KoAT

- 682 examples (including collections of other tools)
- time-out of 60 seconds

Related Work and Experiments

Related Work on Symbolic Complexity Analysis

- PUBS/COSTA (Albert et al)
- Rank (Alias et al)

Prototype Implementation KoAT

- 682 examples (including collections of other tools)
- time-out of 60 seconds
- Time: average runtime on those examples where tool was successful

	1	$\log n$	n	$n \log n$	n^2	n^3	$n^{>3}$	EXP	Time
KoAT	120	0	142	0	59	3	3	0	1.1 s
PUBS	85	4	104	1	13	4	0	6	.3 s
Rank	56	0	19	0	8	1	0	0	.5 s

Related Work and Experiments

Related Work on Symbolic Complexity Analysis

- PUBS/COSTA (Albert et al)
- Rank (Alias et al)

Prototype Implementation KoAT

- 682 examples (including collections of other tools)
- time-out of 60 seconds
- Time: average runtime on those examples where tool was successful

	1	$\log n$	n	$n \log n$	n^2	n^3	$n^{>3}$	EXP	Time
KoAT	120	0	142	0	59	3	3	0	1.1 s
PUBS	85	4	104	1	13	4	0	6	.3 s
Rank	56	0	19	0	8	1	0	0	.5 s

⇒ substantially more powerful and still efficient

Runtime and Size Complexity of Integer Programs

- alternate finding **runtime** and **size** bounds
 - compute **size bounds** by combining local variable changes with **runtime bounds**
 - compute **runtime bounds** for program parts based on **size bounds** for preceding parts
- only consider small program parts at a time
 - linear rank functions usually sufficient
 - approach scales to larger programs
- Extensions
 - recursive procedure calls
 - advanced cost measures
 - separate analysis of procedures and loops

Overall Procedure

Input: Program $\mathcal{T}$

$(\mathcal{R}, \mathcal{S}) := (\mathcal{R}_0, \mathcal{S}_0)$

while there exist t, v with
 $\mathcal{R}(t) = ?$ or $\mathcal{S}(t, v') = ?$ **do**

$\mathcal{T}' := \{t \in \mathcal{T} \mid \mathcal{R}(t) = ?\}$
 $\mathcal{R} := \text{TimeBounds}(\mathcal{R}, \mathcal{S}, \mathcal{T}')$

for all SCCs C of the RVG
in topological order **do**
 $\mathcal{S} := \text{SizeBounds}(\mathcal{R}, \mathcal{S}, C)$
done

done

Output: **Runtime approximation** $\mathcal{R}$,
Size approximation $\mathcal{S}$